The Fractional Order Models of a Thermal Trace on a Heat-Insulating Surface

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Abstract: The paper deals with a modeling of thermal trace at heat insulating surface with the use of two Fractional Order (FO) state space models. The fundamental distributed-parameter model was compared to simplified, lumped-parameter model, built with the assumption that the spatial heat transfer can be omitted. Such a comparison has not been presented yet. The simplifying assumption was confirmed experimentally in two ways. Firstly, the proposed lumped-parameter model assures the same accuracy in the sense of Integral Absolute Error (IAE) cost function as distributed--parameter model. Next, identified values of the heat transfer coefficient in the heat transfer equation are close to zero.

Keywords: fractional order systems, 2D heat transfer, initial problem, Caputo definition, thermal camera, thermal trace

1. Introduction

The modeling of physical phenomena hard to describe using other tools is one of main areas of application of Fractional Order (FO) calculus. Typically, FO models are relatively simple and more accurate in the sense of common cost functions than their Integer Order (IO) analogues.

Such models have been proposed by many authors for years. A number of such models from area of physics and biology can be found e.g. in [4, 33, 34, 14]. The book [3] gives fractional order description of chaotic systems and Ionic Polymer Metal Composites (IPMC).

The "classic" application of FO models is the modeling of ultracapacitors [6]. The use of fractional calculus to modeling diffusion processes is discussed in [7, 30, 32]. A collection of results employing new Atangana-Baleanu operator can be found in [8]. This book deals with i.e. the FO blood alcohol model, the Christov diffusion equation and fractional advection-dispersion equation for groundwater transport process.

Different kinds of heat processes can be also described with the use of FO approach. For example, a temperature-heat flux in semi-infinite conductor is presented in the book [4], the heating of a beam is presented in the paper [6].

The fractional models of the one dimensional heat in the state space has been given in many previous papers of author [18–24].

The two-dimensional generalization of the above models is given in the papers [25, 26]. Each proposed model has been thoroughly theoretically and experimentally verified. In each case the FO model assures the better accuracy than its IO analogue.

Different kinds of temperature trends obtained with the use of thermal imaging cameras can be also described using FO models. This is presented e.g. by [5, 31]. Analytical solution of the two dimensional, integer order heat equation is proposed in the paper [13]. The numerical solving of PDE-s is discussed e.g. in [17]. Fractional Fourier integral operators are analyzed u.a. by [1]. It is important to note that significant part of known investigations deals with steady-state temperature fields with omitting its dynamics.

The modeling of thermal traces left by warm body on cooler ground is an interesting issue from point of view of practical applications in many areas, e.g. in reconstruction of images from thermal camera. The nature of this process points that its model should have the form of a state equation because an initial temperature can be interpreted as an initial condition. Consequently a vanishing of a thermal trace can be expressed by a free response of a model.

The fundamental mathematical model of heat processes in different environments is the Partial Differential Equation (PDE) of the parabolic type, describing a dynamics as well as a spatial distribution of a temperature. The two dimensional, Integer Order (IO) heat transfer equation has been considered in many papers, e.g. [2, 15, 35]. Its fractional version has been proposed and analyzed in [25, 26].

The use of fractional approach in modeling of thermal traces has been proposed only in the paper [27]. The discrete, memory effective FO model presented by it was constructed at once as lumped parameter model. Additionally, this model allows to set an initial condition as a discrete function, describing a behavior of the temperature in the past.

This paper presents two fractional order, time continuous, state space models of a thermal trace. The first one is the fractional order heat transfer equation, the next one is its simplification obtained via elimination of the spatial heat dissipation in heat transfer equation. This can be justified by the fact that

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the modeled thermal trace had been left at wooden laminate. This material is a heat insulator. It allows to ignore the factor describing the heat transmission in the material. This simplification was suggested also by results of identification of the heat transfer equation with the use of experimental data. The identified value of the coefficient of heat transfer in material was close to zero. This is also the powerful justification of simplification assumed in the paper [27].

The paper is organized as follows. Preliminaries recall some elementary ideas necessary to present of the main results. Next the experimental system as well as results of experiments are presented. Furthermore the proposed models of thermal trace are presented, discussed and experimentally validated.

2. Preliminaries

2.1. Elementary ideas

Basics of fractional calculus are presented in many books [4, 9, 28, 29]. Initial problems for discrete FO systems are discussed e.g. in [16]. However the use of the Caputo definition of FO operator allows to use the same initial condition as in IO case. This approach will be employed in this paper.

The fractional order, integro-differential operator is given e.g. in [10]. Its definition is beneath.

Definition 1. *The elementary fractional order operator*
The fractional order integro-differential operator is defined as follows:

$$ {}_aD_t^\alpha f(t) = \begin{cases} \dfrac{d^\alpha f(t)}{dt^\alpha} & \alpha > 0 \\ 1 & \alpha = 0 \\ \displaystyle\int_a^t f(\tau)(d\tau)^{-\alpha} & \alpha < 0 \end{cases} \quad (1)$$

where a and t denote time limits of operator calculation, $\alpha \in \mathbb{R}$ *denotes the fractional order of the operation.*

Next an idea of Gamma Euler function is recalled (see for example [12]):

Definition 2. *The Gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \quad (2)$$

Next recall an idea of Mittag-Leffler function. It is a non-integer order generalization of exponential function $e^{\lambda t}$ and it is applied to solve a fractional order state equation. The one parameter Mittag-Leffler function is defined as follows:

Definition 3. *The one parameter Mittag-Leffler function*

$$E_\alpha(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(k\alpha + 1)}. \quad (3)$$

The fractional-order, integro-differential operator is expressed by different definitions. The most known are given by Grünvald and Letnikov (this is so called GL definition), Riemann and Liouville (RL definition) and Caputo (C definition). In this paper only C definition is applied. Its use allows to express linear state equation analogically, as for IO case. In addition, its application allows to define an initial condition analogically as in IO case. The C definition is given e.g. in [11]. It is as follows:

Definition 4. *The Caputo definition of the FO operator*

$${}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(L-\alpha)} \int_0^\infty \frac{f^{(L)}(\tau)}{(t-\tau)^{\alpha+1-L}} d\tau, \quad (4)$$

where $L - 1 < \alpha < L$ *is the non-integer order of an operation and* $\Gamma(..)$ *is the complete Gamma function* (2).

A fractional-order linear, free, SISO, state space system, employing C definition is described by the following state equation:

$$\begin{aligned} {}_0^CD_t^\alpha f(t) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned} \quad (5)$$

where $\alpha \in (0,1)$ denotes the fractional order of the state equation, $x(t) \in \mathbb{R}^N$, $Y(t) \in \mathbb{R}$ are the state and output vectors respectively, $A_{N\times N}$ and $C_{1\times N}$ are state and output matrices. For this system only its free response will be analyzed, but this is sufficient to modeling of thermal trace.

3. The experimental system

The simple experimental system is shown in the Fig. 1. This is flat surface of the table with thermal camera attached vertically over table. The table top is made of wood laminate coated with plastic. This material is good heat insulator and this determines the properties of a model describing thermal processes occuring on it.

The dimension of measured area is determined by the size of the sensor and the focal length of camera's lens. In the experiment the camera OPTRIS PI 450 with lens O29 29° × 22° was applied. The resolution of the camera's sensor is 382 × 288 pixels. The camera is attached 300 mm over table and the applied lens gives field of view 165 mm × 121 mm with the size of the single pixel equal 0.43 mm × 0.42 mm. The diameter of the trace is equal: $\phi = 75$ mm. Data from camera are read using dedicated software OPTRIS PI Connect.

The experiment consisted of placing a hot cup on the table and taking it away after a while. Thermal trace left by the cup was measured by K time moments. The first measurement is used as the initial condition. The 3D temperature

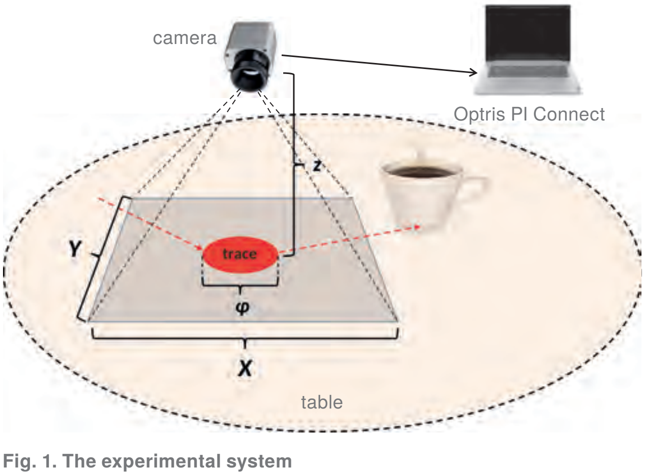

Fig. 1. The experimental system
Rys. 1. System doświadczalny

distributions for initial and final time instants are illustrated by the Fig. 2. The X and Y coordinates are given in [mm], the temperature is given in Celsius degrees. The contours of the same temperature distributions and places of time trends reading are shown in the Fig. 3. To facilitate the indication of measurement points, the coordinates are given in pixels. The time trends of temperature in two exemplary points are shown in the Fig. 4. It is important to note that measurements with thermal camera can be affected by various disturbations, e.g. by varying and unknown emissivity of the surface, light reflections and by random ambient temperature.

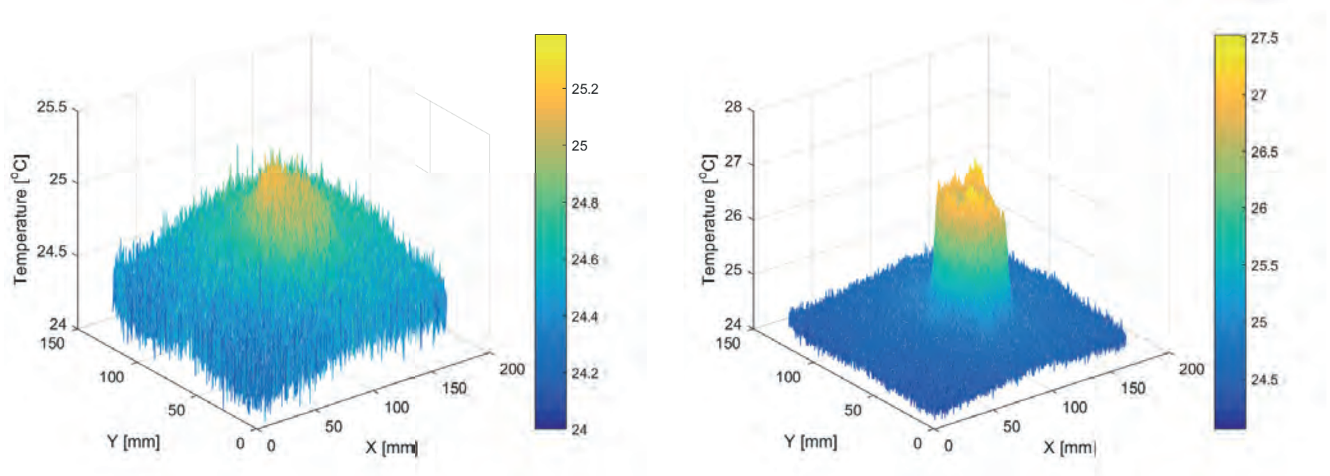

Fig. 2. 3D diagrams of thermal traces for initial (left) and final (right) time instants

Rys. 2. Wykresy trójwymiarowe śladu termicznego w chwili początkowej (lewa) i końcowej (prawa)

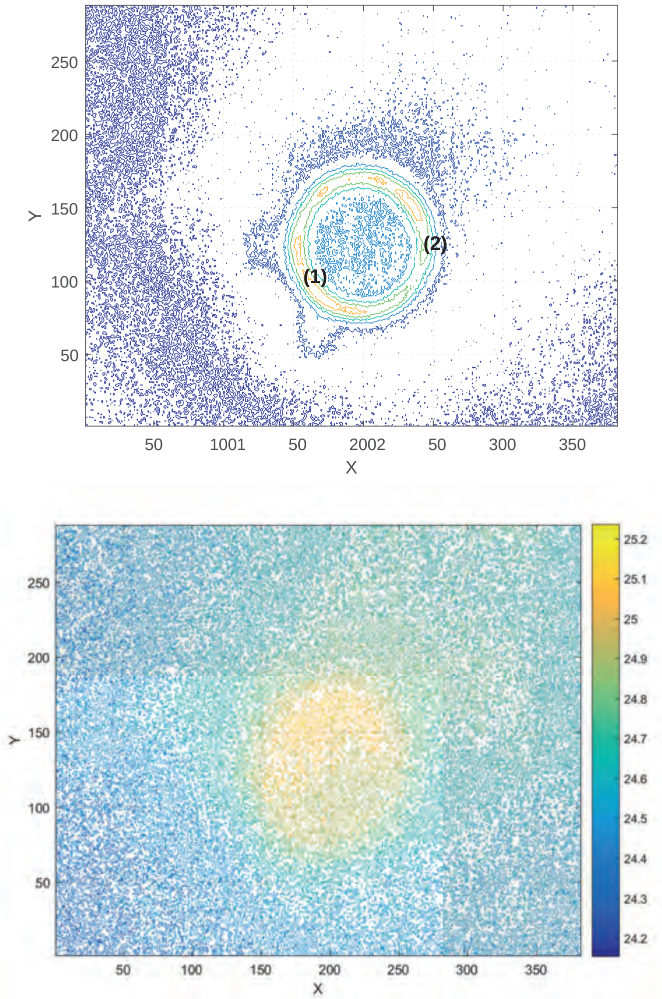

Fig. 3. Contours of thermal traces for initial (top) and final (bottom) time instants. At the initial contour are marked the points, where time trends were measured (dimensions X and Y in pixels)

Rys. 3. Wykresy poziomicowe śladu termicznego dla chwili początkowej (góra) i końcowej (dół). Na wykresie początkowym zaznaczono punkty pomiaru trendów czasowych (wymiary X i Y w pikselach)

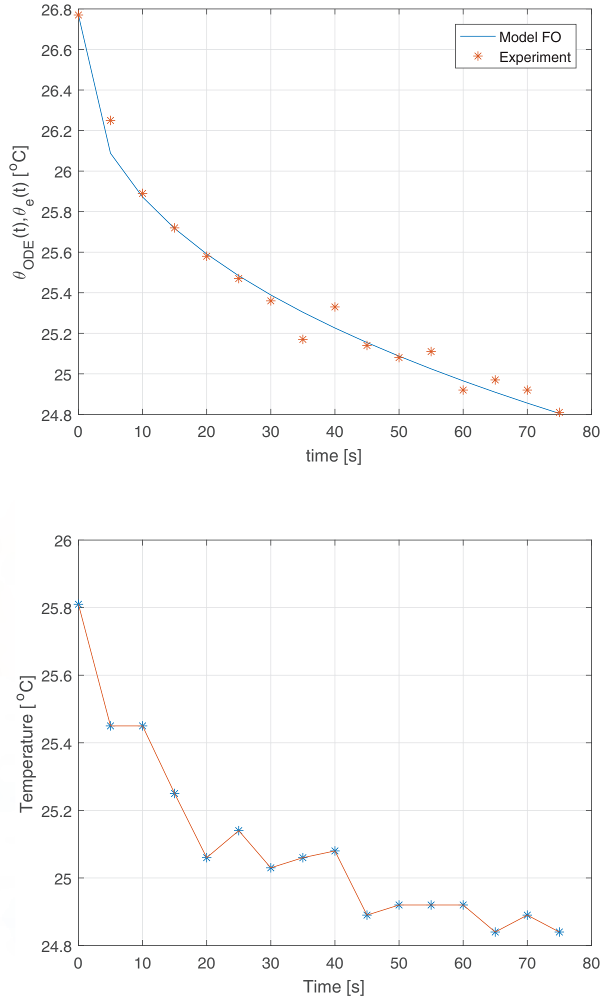

Fig. 4. Time trends of temperature in points: 1: (165,100) – top, 2: (250,125) – bottom

Rys. 4. Trendy czasowe temperatury w zaznaczonych punktach

4. Main results

4.1. The free, FO, partial differential equation

The temperature of the plate is determined by the heat transfer and dissipation. The exact model of this process, describing both time and distance dependencies take the form of a partial differential equation. To improve its accuracy the fractional derivatives along time and space can be employed. This model is analysed with details in the paper [26]. Here its free version (without control) will be used. It is presented beneath.

The temperature is measured using thermal camera, the area of measurement is configurable and denoted by S. Its coordinates are equal x_{s1}, x_{s2}, y_{s1} and y_{s2}. The surface area S_S of the measurement area is equal:

$$S_S = d_{xs} d_{ys} \tag{6}$$

where:

$$\begin{aligned} d_{xs} &= x_{s2} - x_{s1}, \\ d_{yh} &= y_{s2} - y_{s1}. \end{aligned} \tag{7}$$

The heat transfer in the surface is described by the Partial Differential Equation (PDE) of the parabolic type. All the side surfaces of plate are much smaller than its frontal surface. This allows to assume the homogeneous Neumann boundary conditions at all edges of the plate as well as the heat exchange on the surface needs to be also considered. It is expressed by coefficient R_a. The control and observation are distributed due to the size of heater and size of temperature field read by camera. The spatial heat conduction along both directions x and y is the same and it is described by coefficient a_w.

The two dimensional, IO heat transfer equation has been considered in many papers [2, 15, 35]. Its fractional version with fractional derivative along the time and 2nd order integer derivative along the length is presented with details in the paper [25].

The proposed model, fully describing the considered thermal process takes the following form:

$$\begin{cases} {}_0^C D_t^{\alpha} Q(x,y,t) = a_w \left(\dfrac{\partial^{\beta} Q(x,y,t)}{\partial x^{\beta}} + \dfrac{\partial^{\beta} Q(x,y,t)}{\partial y^{\beta}} \right) - R_a Q(x,y,t) \\ \dfrac{\partial Q(0,y,t)}{\partial x} = 0 \quad t \ge 0 \\ \dfrac{\partial Q(X,y,t)}{\partial x} = 0 \quad t \ge 0 \\ \dfrac{\partial Q(x,0,t)}{\partial y} = 0 \quad t \ge 0 \\ \dfrac{\partial Q(x,Y,t)}{\partial y} = 0 \quad t \ge 0 \\ Q(x,y,0) = 0 \quad t \ge 0 \\ \theta(t) = k_0 \int_0^X \int_0^Y Q(x,y,t) c(x,y) dx dy \end{cases} \tag{8}$$

In (8) α and β are non-integer orders of the system, $a_w > 0$, $R_a \in \mathbb{R}$ are coefficients of heat conduction and heat exchange, k_0 is a steady-state gain of the model.

The heat exchange at the borders of the plate is described by the homogenous Neuman boundary conditions. This is justified by the fact that the heat exchange through a side surfaces is negligibly small compared to heat exchange through the front surface of a plate.

Finally, $c(x, y)$ is the sensor function in the following form:

$$c(x,y) = \begin{cases} 1, x,y \in S \\ 0, x,y \notin S \end{cases}. \tag{9}$$

4.2. The free, FO, state equation

The equation (8) can be expressed as an infinite dimensional state equation (details – see [26]). It has the form (5):

$$\begin{cases} {}_0^C D_t^{\alpha} Q(t) = A Q(t) \\ \theta(t) = C Q(t) \end{cases}. \tag{10}$$

where:

$$\begin{aligned} & AQ = a_w \left(\frac{\partial^{\beta} Q(x,y)}{\partial x^{\beta}} + \frac{\partial^{\beta} Q(x,y)}{\partial y^{\beta}} \right) - R_a Q(x,y), \\ & D(A) = \{ Q \in H^2(0,1) : \quad Q'(0) = 0, \quad Q'(X) = 0, \quad Q'(Y) = 0 \}, \\ & a_w > 0, \quad R_a > 0, \\ & CQ(t) = \langle c, Q(t) \rangle . \end{aligned} \tag{11}$$

The state vector $Q(t)$ is defined as beneath:

$$Q(t) = \left[q_{0,0}, q_{0,1}, q_{0,2}, \ldots, q_{1,1}, q_{1,2}, \ldots \right]^{\mathrm{T}}. \tag{12}$$

The eigenfunctions and eigenvalues of the state operator A take the following form:

$$w_{m,n}(x,y) = \begin{cases} 1, & m = 0, n = 0 \\ \dfrac{2Y}{\pi n} \cos \dfrac{n \pi y}{Y}, & m = 0, n = 1,2,\ldots \\ \dfrac{2X}{\pi m} \cos \dfrac{m \pi x}{X}, & n = 0, m = 1,2,\ldots \\ \dfrac{2}{\pi} \dfrac{1}{\sqrt[\beta]{\dfrac{m^{\beta}}{X^{\beta}} + \dfrac{n^{\beta}}{Y^{\beta}}}} \cos \dfrac{m \pi x}{X} \cos \dfrac{n \pi y}{Y}, & m = 1,2,\ldots \; n = 1,2,\ldots \end{cases} \tag{13}$$

$$\lambda_{m,n} = -a_w \left[\frac{m^{\beta}}{X^{\beta}} + \frac{n^{\beta}}{Y^{\beta}} \right] \pi^{\beta} - R_a. \quad m = 0,1,2,\ldots \; n = 0,1,2,\ldots \tag{14}$$

The state operator A takes the following form:

$$A = \operatorname{diag} \left\{ \lambda_{0,0}, \lambda_{0,1}, \lambda_{0,2}, \ldots, \lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{2,1}, \lambda_{2,2}, \ldots, \lambda_{m,n}, \ldots \right\}. \tag{15}$$

The output (observation) operator C is as beneath:

$$C = \left[c_{0,0}, c_{0,1}, \ldots, c_{1,0}, c_{1,1}, \ldots \right]. \tag{16}$$

where:

$$c_{m,n} = \langle S, w_{m,n} \rangle = \int_0^X \int_0^Y c(x,y) w_{m,n}(x,y) dx dy. \tag{17}$$

Calculating of the integral (17) yields:

$$c_{m,n} = \begin{cases} S_S, & m=0,\ n=0 \\ \frac{2Y^2}{h_{yn}^2} d_{xs} a_{nsy}, & m=0,\ n=1,2,3,\ldots \\ \frac{2X^2}{h_{xm}^2} d_{ys} a_{msx}, & n=0,\ m=1,2,3,\ldots \\ \frac{k_{m,n}}{h_{xm} h_{yn}} a_{msx} a_{nsy}, & m=1,2,3,\ldots\ n=1,2,3,\ldots \end{cases} \tag{18}$$

where:

$$h_{xm} = \frac{m\pi}{X}, \quad h_{yn} = \frac{n\pi}{Y}. \tag{19}$$

$$k_{m,n} = \frac{2}{\pi} \frac{1}{\sqrt[\beta]{\frac{m^\beta}{X^\beta} + \frac{n^\beta}{Y^\beta}}}. \tag{20}$$

$$a_{mhx} = \sin\left(h_{xm} x_{h2}\right) - \sin\left(h_{xm} x_{h1}\right), \quad a_{nhy} = \sin\left(h_{yn} y_{h2}\right) - \sin\left(h_{yn} y_{h1}\right). \tag{21}$$

$$a_{msx} = \sin\left(h_{xm} x_{s2}\right) - \sin\left(h_{xm} x_{s1}\right), \quad a_{nsy} = \sin\left(h_{yn} y_{s2}\right) - \sin\left(h_{yn} y_{s1}\right). \tag{22}$$

The model (8)(22) is infinite dimensional. Its application to modeling of a heat transfer requires to use its finite dimensional approximation. Such an approximation is easy to propose due to the system can be decomposed to infinite number of independent modes $q_{m,n}$ [26]. Consequently to modeling only the finite number $M * N$ of modes $q_{m,n}(t)$ can be used with omitting the others. In this situation the operators A and C are interpreted as matrices. The size $M * N$ of the finite dimensional model can be estimated numerically or analytically (see [26]).

The free response of the finite dimensional system takes the following form:

$$\theta_{PDE}(t) = k_s \theta_0 \sum_{m,n=0}^{M,N} c_{m,n} q_{m,n}(t). \tag{23}$$

where k_s is the steady-state gain of the model, θ_0 is the initial temperature, $c_{m,n}$ are expressed by (18) and the mode $q_{m,n}(t)$ takes the following form:

$$q_{m,n}(t) = E_\alpha\left(\lambda_{m,n} t^\alpha\right). \tag{24}$$

In (24) $E_\alpha(\ldots)$ is the one parameter Mittag-Leffler function. The equation (23) describes the behaviour of the thermal trace in time and space. Its complexity is relatively high: for example the dimensions $M = N = 7$ give the size of the whole model equal 49.

4.3. The free, FO, scalar state equation

The substrate on which we test thermal traces is a wood laminate coated with plastic. Such a material is a good heat insulator. This makes it possible to omit spatial heat dissipation in the material. Consequently the coefficient $a_w = 0.0$ and the partial equation (8) reduces to the ordinary differential equation:

$$\begin{cases} {}_0^C D_t^\alpha Q\left(x,y,t\right) = -R_a Q\left(x,y,t\right), \\ \theta(t) = CQ(t). \end{cases} \tag{25}$$

Consequently the solution of (25) takes the following form:

$$\theta_{ODE}(t) = \theta_0 E_\alpha\left(-R_a t^\alpha\right). \tag{26}$$

5. Experimental validation of results

To verify results presented in the previous section the experimental system shown in the fig. 1 was used. Initial and final temperature fields as well as temperature trends in examined points are shown in Figures 2 and 4. The temperature was measured with the sample time $h = 5$ s, the amount of measurements was equal: $K = 16$.

Estimation of a performance of different models by comparing of time trends is not enough accurate. Such a comparison requires to use of a cost function. Additionally such a cost function can be employed to parameters identification. In this case the Integral Absolute Error (IAE) cost function was employed. It describes the absolute value of difference between free responses of plant and model at the same time grid. It is an implicit, complex function of the model parameters: α, β, a_w and R_a:

$$IAE_{PDE,ODE} = h\sum_{k=1}^{K}\left|\theta_{PDE,ODE}(k) - \theta_e(k)\right|. \tag{27}$$

In (27) h is the sample time, K is the number of measurements, $\theta_{PDE,ODE}(k)$ are the analytical free responses of the tested PDE and ODE models respectively. They are computed using equations (23) or (26) evaluated at the MATLAB platform at the same time grid, as experiment.

Table 1. Parameters of the distributed parameter model (10)
Tabela 1. Parametry modelu o parametrach rozłożonych M = N

x	y	M, N	α	β	a_w	R_a	IAE (27)
165	100	3	0.4016	2.9616	3.7560E4	0.0115	0.7587
165	100	5	0.4729	2.166	0.0001	0.0031	0.7222
165	100	7	0.4246	2.7166	9.01E-5	0.0105	0.7558
250	125	3	0.3587	1.9436	4.49E-4	0.0071	0.6576
250	125	5	0.3571	1.8312	0.0001	0.0083	0.6541
250	125	7	0.5193	2.0666	0.0001	0.0015	0.7383

Table 2. Parameters of the lumped parameter model (26)
Tabela 2. Parametry modelu o parametrach skupionych (26)

x	y	α	R_a	*IAE* (27)
165	100	0.4066	0.0120	0.7572
250	125	0.3587	0.0075	0.6631

The parameters of both models were found via minimization of the cost function (27) with the use of MATLAB function *fminsearch*. The identified parameters of the PDE model are given in the Table 1 and the identified parameters of the ODE model are given in the Table 2.

6. Discussion of results

At the beginning it is important to note that the best accuracy of the PDE model was achieved for orders M, N equal 5 and it decreases for orders equal 7.

Next the detailed comparing of the cost function *IAE* for both models (23) and (26) should be done. It is illustrated by the table 3. For the PDE model its most accurate version obtained for $M = N = 5$ is considered. The right column of this table presents the relative difference between IAE_{PDE} vs IAE_{ODE} calculated as follows:

$$IAE_{\Delta} = \frac{\left|IAE_{PDE} - IAE_{ODE}\right|}{IAE_{PDE}} 100\%. \tag{28}$$

The Table 3 shows that the maximum difference between accuracy of the model PDE vs ODE does not exceed 5 %.

Furthermore, the identified value of the heat transfer coefficient a_w in the parabolic equation (8) is close to zero for all tested places and for each tested dimensions of the PDE model, expressed by M. This is as expected because the tested surface is good heat insulator.

Finally, the use of the simpler model (25) is justified by the material properties of the substrate as well as the identification results of both models.

Table 3. The values of the cost function (27) for both examined models and their relative difference IAE_{Δ}
Tabela 3. Wartości funkcji kosztu (27) dla obu testowanych modeli i ich względna różnica IAE_{Δ}

x	y	*IAE* (27) for PDE (23)	*IAE* (27) for ODE (26)	IAE_{Δ} (28) in %
165	100	0.7222	0.7572	4.85
250	125	0.6541	0.6631	1.38

7. Final conclusions

The main final conclusion from the paper is that the modeling of thermal traces left at heat insulating substrates does not require to use of complex models in the form of partial

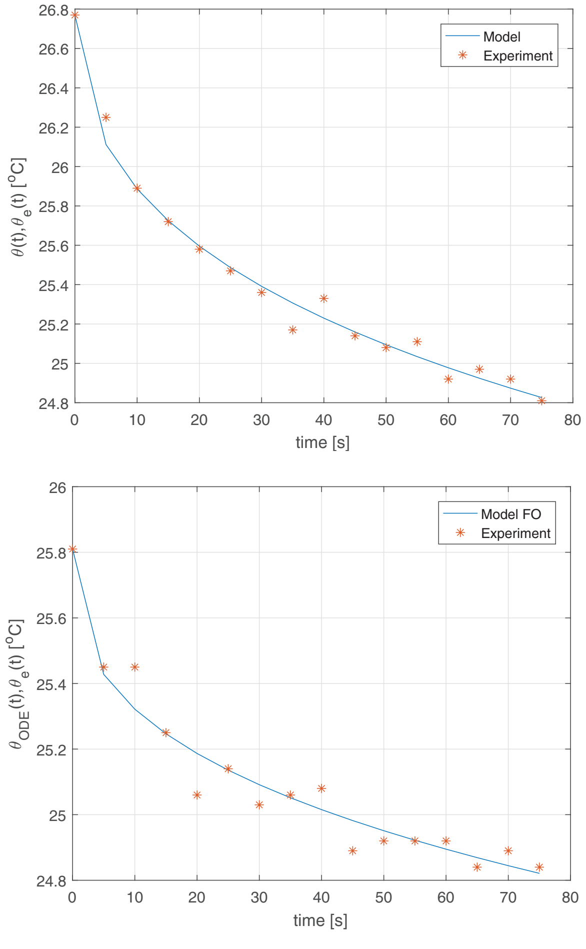

Fig. 5. Comparison of responses of PDE model vs experiment for points: (165,100) – top, (250,125) – bottom
Rys. 5. Porównanie odpowiedzi modelu PDE z eksperymentem dla punktów: (165,100) – góra, (250,125) – dół

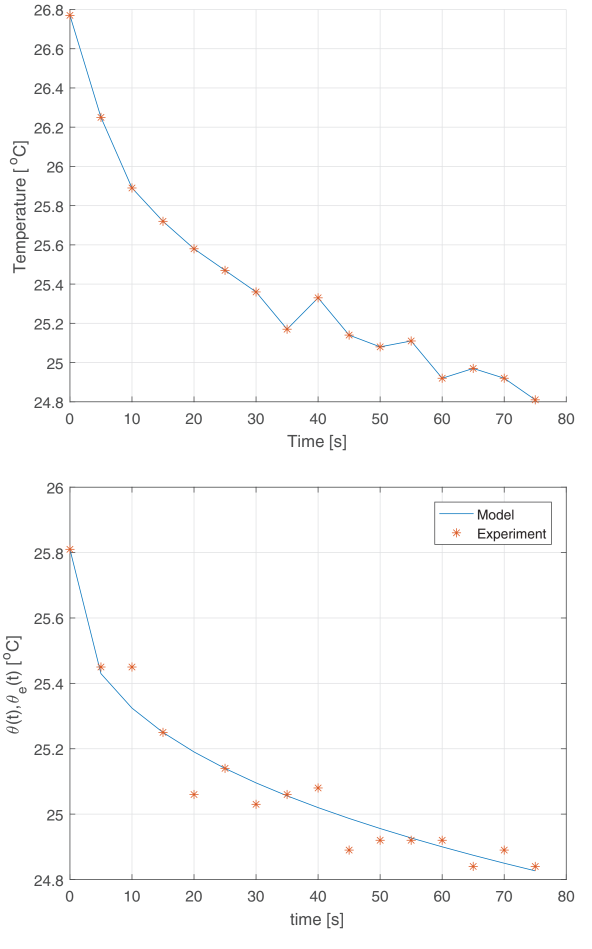

Fig. 6. Comparison of responses of ODE model (25) vs experiment for points: (165,100) – top, (250,125) – bottom
Rys. 6. Porównanie odpowiedzi modelu ODE z eksperymentem dla punktów: (165,100) – góra, (250,125) – dół

differential equations. The ordinary differential equation is sufficient. It assures practically the same accuracy and it is easier to identify and use.

The further investigations of the presented issue will cover for example an application of other definitions of fractional operator and its implementation at different digital platforms.

An another interesting issue is a response to question, for which materials the use of a PDE model is necessary and when an ODE model is sufficient.

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Modele ułamkowe rzędu śladu termicznego na powierzchni izolującej ciepło

Streszczenie: W pracy omówiono zagadnienie modelowania śladu termicznego na powierzchni izolującej ciepło z wykorzystaniem modeli ułamkowego rzędu w przestrzeni stanu. Podstawowy model o parametrach rozłożonych porównano z jego uproszczeniem o parametrach skupionych, zbudowanym przy założeniu, że przestrzenne rozchodzenie się ciepła w materiale płyty może być pominięte. Takie porównanie modelu o parametrach rozłożonych z modelem o parametrach skupionych nie było dotąd prezentowane. Założenie upraszczające zostało potwierdzone doświadczalnie dwiema niezależnymi drogami. Po pierwsze, dokładność (w sensie wskaźnika jakości IAE) modelu o parametrach skupionych jest praktycznie taka sama, jak modelu o parametrach rozłożonych. Po drugie, wartości współczynnika przewodnictwa cieplnego otrzymane w wyniku identyfikacji modelu są bliskie zera.

Słowa kluczowe: system rzędu ułamkowego, dwuwymiarowe przewodnictwo cieplne, problem początkowy, definicja Caputo, kamera termowizyjna, ślad termiczny

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