Many Faces of Singularities in Robotics

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Abstract: In this survey paper some issues concerning a singularity concept in robotics are addressed. Singularities are analyzed in the scope of inverse kinematics for serial manipulator, a motion planning task of nonholonomic systems and the optimal control covering a large area of practical robotic systems. An attempt has been made to define the term singularity, which is independent on a specific task. A few classifications of singularities with respect to different criteria are proposed and illustrated on simple examples. Singularities are analyzed from a numerical and physical point of view. Generally, singularities pose some problems in motion planning and/or control of robots. However, as illustrated on the example on force/momenta transformation in serial manipulators, they can also be desirable is some cases. Singularity detection techniques and some methods to cope with them are also provided. The paper is intended to be didactic and to help robotic researchers to get a general view on the singularity issue.

Keywords: holonomic systems, nonholonomic systems, singularities, classification, detection, avoiding

1. Introduction

Singularities appear in many contexts in robotics and usually cause some problems in motion planning and/or controlling a robot. In this paper singularities are analyzed for different types of robots: mobile/free floating (nonholonomic ones) as well as for holonomic stationary manipulators. In the most common case, singularities arise when some matrices used to plan or control a robot motion lose their full admissible rank and problems of their (generalized) inversion arise. Two aspects are to be mentioned: from a geometrical point of view at singularities motions in certain directions are not permitted, from a numeric perspective either algorithms stop to work or their results are unreliable. Singularities occupy a relatively small part of a space where they are defined. However, their close neighborhood is quite massive and singularity disregarding may cause a numerical instability. Moreover, around singularities some characteristics (like velocities at joints) can take inadmissible, too large values and/or they switch their signs causing undesirable chattering.

On the other hand singularities result either due to improper modeling of a robot or its description is not valid globally. In order to search for a unifying definition of singularities one needs to consult their linguistic meaning. From a point of view of applied sciences probably the most suitable notion of singularity is covered by its equivalent terms: strange, un-regular, and exceptional. It means that at singularities something unusual happens and this case cannot be treated in a standard (regular) way. From a practical point of view at singularities either a description of the controlled object should be updated or changed or algorithms designed for a regular case modified slightly or substantially. In any case special actions for the singular situation should be foreseen before starting planning or control and while performing regular actions a permanent monitoring should take place to react properly on singular cases. The singularity analysis still attracts attention not only roboticians but also mathematicians [3].

This paper, being extended version of the conference paper [7], is organized as follows. In Section 2 singularities are described in three areas of robotics: for manipulators representing holonomic systems exemplified on the task of inverse kinematics, for nonholonomic systems performing a motion planning task and quite general systems to be controlled optimally. In Section 3 some methods to detect singularities are highlighted while in Section 4 techniques to cope with them are presented. Section 5 concludes the paper.

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2. Types of singularities

2.1. Holonomic systems

A standard kinematics for serial, single chain open loop manipulators is given as

$$\boldsymbol{k}:\ \mathbb{Q} \ni \boldsymbol{q} \to \boldsymbol{k}(\boldsymbol{q}) \in \mathbb{X},\ \dim \mathbb{Q} = n,\ \dim \mathbb{X} = m. \tag{1}$$

For a goal point in the task-space $\boldsymbol{x}_f \in \mathbb{X}$ an analytic inverse of (1) can be found only for a few manipulators. Therefore the Newton algorithm is used, below given in the form valid for redundant, $n > m$ case (for non-redundant manipulators $\boldsymbol{J}^{\#}$ is substituted with the standard matrix inversion $\boldsymbol{J}^{-1}$) [16]

$$\boldsymbol{q}_{i+1} = \boldsymbol{q}_i + \xi_i \cdot \boldsymbol{J}^{\#}\left(\boldsymbol{q}_i\right)\left(\boldsymbol{x}_f - \boldsymbol{k}\left(\boldsymbol{q}_i\right)\right). \qquad (2)$$

The initial configuration $\boldsymbol{q}_0$ for the iterative process (2) is given and the pseudo-inverse $\boldsymbol{J}^{\#}$ of the matrix $\boldsymbol{J}$ is defined via the manipulability matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q}) \quad \text{as} \quad \boldsymbol{J}^{\#} = \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})^{-1}. \qquad (3)$$

Sometimes either to increase/decrease importance of some coordinates or to unify units (length-angle coordinates for prismatic-rotational joints) the weighted-version of pseudo-inverse is used

$$\boldsymbol{J} \quad = \boldsymbol{W}^{-1}\boldsymbol{J}^{\mathrm{T}}\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^{\mathrm{T}}\right)^{-1}, \qquad (4)$$

where $\boldsymbol{W}$ denotes a symmetric, positively definite weighting matrix. In any case, to effectively compute inverse (non-redundant manipulators) or (weighted-) pseudoinverse (redundant manipulators) matrix, a square matrix has to be non-singular.

The kinematics is exemplified on the 2D planar pendulum, depicted in Fig. 1, with positional kinematics

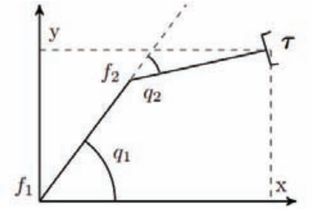

Fig. 1. The planar double pendulum: configuration (q_1, q_2), task-space coordinates (x, y), end-effector forces τ and forces at joints (f_1, f_2) depicted
Rys. 1. Podwójne wahadło planarne: konfiguracja (q_1, q_2), współrzędne przestrzeni zadaniowej (x, y), siły na efektorze τ i w przegubach (f_1, f_2)

$$\boldsymbol{k}(\boldsymbol{q}) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \end{bmatrix}, \qquad (5)$$

where a_1, a_2 are lengths of its links and a simplified form to denote trigonometric functions, $c_{12} = \cos(q_1 + q_2)$, $s_1 = \sin(q_1)$, is used. The Jacobian matrix of the 2D pendulum is given as

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{k}}{\partial \boldsymbol{q}} = \begin{bmatrix} -a_1s_1 - a_2s_{12} & -a_2s_{12} \\ a_1c_1 + a_2c_{12} & a_2c_{12} \end{bmatrix}. \qquad (6)$$

From the determinant condition

$$\det(\boldsymbol{J}(\boldsymbol{q})) = a_1\, a_2\, s_2 = 0$$

a singular configuration sub-space is derived

$$(q_1,\ q_2) = (\star,\ \{0, \pi\}), \qquad (7)$$

where, here and afterwards, $\star$ denotes any value.

Here we can distinguish length and angle singularities. The former ones are due to vanishing of some rows of the Jacobi matrix (at the configuration $\boldsymbol{q} = (0,\ 0)^{\mathrm{T}}$ ($\boldsymbol{q} = (\pi/2,\ 0)^{\mathrm{T}}$) the length of the first (second) row in (6) is equal to zero. The latter ones arise when a row becomes a linear combination of other rows (the exemplary singular configuration $(\pi/4,\ 0)^{\mathrm{T}}$). This characteristics can be expressed as angle dependencies, as some row-vectors of the Jacobi matrix are placed on a single hyperplane and some angle relationships between them hold. It can be noticed that the length singular configurations are not typical (for the 2D planar pendulum they appear only at configurations $\left(r\pi/2, \{0, \pi\}\right)^{\mathrm{T}}$, $r \in \mathbb{N}$), cf. Fig. 2 forming a set composed of isolated points. Theoretically, length singularities of corank s arise when simultaneously $s \cdot n$ equality conditions hold, i.e. s rows of the Jacobi matrix vanish. In practice s can be equal to 1 only as at most n independent constraints can be imposed on the configuration space and the constraints are satisfied on a 0-dimensional manifold (separated points). Consequently, the singularities can not be typical at all.

Singular configurations can be classified based on characteristics they depend on. For earth-based stationary manipulators kinematic singularities can be distinguished (depending, beside configurations, on lengths and twists of links) while for free-floating space manipulators dynamic singularities appear as they depend also on masses and inertia parameters of robots. The aforementioned differentiation may be slightly deceptive as kinematic singularities can be assigned as well to problems related to kinematics while dynamic ones to those involved dynamics of robots.

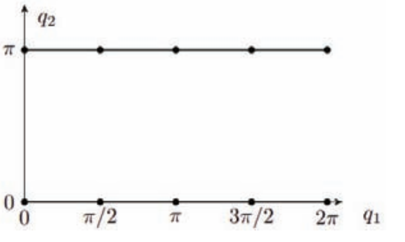

Fig. 2. Singular configurations of 2D planar pendulum: dots – length singularities, thick lines – angle singularities
Rys. 2. Konfiguracje osobliwe dwuwahadła planarnego w przestrzeni konfiguracyjnej: kropki – osobliwości długościowe, linia pogrubiona – osobliwości kątowe

The latter division can be even more complicated as for many systems of robotics their kinematics is directly incorporated into dynamics either in a motion planning within the task-space or at a control level in a cascade kinematic-dynamic scheme.

For any manipulator with a high dimensional, $m = 6$, task-space parameterization singularities arise due to the minimal representation of a rotational group $\mathbb{SO}(3)$. A (3×3) rotational matrix

$$\boldsymbol{R} = \left[\boldsymbol{n}\ \boldsymbol{o}\ \boldsymbol{a}\right] \in \mathbb{SO}(3), \qquad (8)$$

composed of columns $\boldsymbol{n}$, $\boldsymbol{o}$, $\boldsymbol{a}$ is a nine dimensional object with six independent constraints imposed

$$\|\boldsymbol{n}\| = \|\boldsymbol{o}\| = \|\boldsymbol{a}\| = 1, \quad \boldsymbol{n} \times \boldsymbol{o} = \boldsymbol{a}, \qquad (9)$$

where $\times$ denotes a cross product and $\|\cdot\|$ is an Euclidean metrics. Consequently $\dim(\mathbb{SO}(3))9 - 6 = 3$ and theoretically three variables are enough to describe any rotation matrix $\boldsymbol{R}$. Unfortunately, there is no global diffeomorphism between $\mathbb{SO}(3)$ and $\mathbb{R}^3$ (or $\mathbb{S}^3$). The most common task of robotics, the inverse kinematics, can be effectively solved only at a velocity level, thus some kind of angular velocity should be considered. The angular velocity in the space frame is derived from the formula $[\omega] = \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}$ [15] and for two exemplary x-y-x and z-y-x parameterizations of $\boldsymbol{R} \in \mathbb{SO}(3)$ with angles $\boldsymbol{p} = (\alpha, \beta, \gamma)^{\mathrm{T}}$ is described by the following matrices

$$\boldsymbol{R}_{xyx} = \mathrm{rot}(x,\alpha)\,\mathrm{rot}(y,\beta)\,\mathrm{rot}(x,\gamma) = \begin{bmatrix} c_\beta & s_\beta s_\gamma & s_\beta c_\gamma \\ s_\alpha s_\beta & c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma & -s_\alpha c_\beta c_\gamma - c_\alpha s_\gamma \\ -c_\alpha s_\beta & s_\alpha c_\gamma + c_\alpha c_\beta s_\gamma & c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma \end{bmatrix},$$

$$\boldsymbol{R}_{zyx} = \mathrm{rot}(z,\alpha)\,\mathrm{rot}(y,\beta)\,\mathrm{rot}(x,\gamma) = \begin{bmatrix} c_\alpha c_\beta & -c_\gamma s_\alpha + c_\alpha s_\beta s_\gamma & c_\alpha c_\gamma s_\beta + s_\alpha s_\gamma \\ c_\beta s_\alpha & c_\alpha c_\gamma + s_\alpha s_\beta s_\gamma & c_\gamma s_\alpha s_\beta - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}, \tag{10}$$

(where rot($axis$, $angle$) denotes an elementary rotation [19]) resulting in angular velocities

$$\boldsymbol{\omega}_{xyx} = \begin{bmatrix} 1 & 0 & c_\beta \\ 0 & c_\alpha & s_\alpha c_\beta \\ 0 & s_\alpha & -c_\alpha s_\beta \end{bmatrix} \dot{\boldsymbol{p}} = \boldsymbol{A}_{xyx}\dot{\boldsymbol{p}},$$

$$\boldsymbol{\omega}_{zyx} = \begin{bmatrix} 0 & -s_\alpha & c_\alpha c_\beta \\ 0 & c_\alpha & s_\alpha c_\beta \\ 1 & 0 & -s_\beta \end{bmatrix} \dot{\boldsymbol{p}} = \boldsymbol{A}_{zyx}\dot{\boldsymbol{p}}. \tag{11}$$

As $\boldsymbol{A}$ matrices should be inverted to work within a task space including coordinates $\boldsymbol{p}$,

$$\dot{\boldsymbol{p}} = \boldsymbol{A}^{-1}_{xyx}\boldsymbol{\omega}_{xyx} = \frac{1}{s_\beta}\begin{bmatrix} s_\beta & -c_\beta s_\alpha & c_\alpha c_\beta \\ 0 & c_\alpha s_\beta & s_\alpha s_\beta \\ 0 & s_\alpha & -c_\alpha \end{bmatrix}\boldsymbol{\omega}_{xyx},$$

$$\dot{\boldsymbol{p}} = \boldsymbol{A}^{-1}_{zyx}\boldsymbol{\omega}_{zyx} = \frac{1}{c_\beta}\begin{bmatrix} c_\alpha s_\beta & c_\alpha s_\beta & c_\beta \\ -s_\alpha c_\beta & c_\alpha c_\beta & 0 \\ c_\alpha & s_\alpha & 0 \end{bmatrix}\boldsymbol{\omega}_{zyx} \tag{12}$$

then for a pair of angles $\boldsymbol{\beta}$ parameterization singularities are encountered

$$\det\left(\boldsymbol{A}_{xyx}\right) = -s_\beta \Rightarrow \beta = \{0,\pi\},$$

$$\det\left(\boldsymbol{A}_{zyx}\right) = -c_\beta \Rightarrow \beta = \{\pi/2,-\pi/2\}, \tag{13}$$

which correspond to rotation matrices at the singularities

$$\boldsymbol{R}_{xyx}(0,\pi) = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & c_{\alpha\pm\gamma} & \mp s_{\alpha\pm\gamma} \\ 0 & s_{\alpha\pm\gamma} & \pm c_{\alpha\pm\gamma} \end{bmatrix},$$

$$\boldsymbol{R}_{zyx}(\pi/2,-\pi/2) = \begin{bmatrix} 0 & -s_{\alpha\mp\gamma} & \pm c_{\alpha\mp\gamma} \\ 0 & c_{\alpha\mp\gamma} & \pm s_{\alpha\mp\gamma} \\ \mp 1 & 0 & 0 \end{bmatrix} \tag{14}$$

respectively. From Eq. (14), it can be deduced that at singular configurations only a sum/difference of angles α, γ can be determined uniquely. At regular configurations two sets of angles (α, β, γ) correspond to a given rotation matrix. Moreover, the aforementioned derivations and observations are also valid for all other three element parameterization of $\mathbb{SO}(3)$ having singularities at different locations within the space.

To avoid parameterization singularities a redundant, four dimensional, parameterization of the $\mathbb{SO}(3)$ group with quaternions is used frequently. Another method of avoiding problems with parameterization singularities is to switch parameterizations as matrices $\boldsymbol{R}$ at singularities for different parameterizations are firmly well separated, cf. Eq. (14), [4].

Singular configurations are usually detected by a drop of the maximal allowable rank of a Jacobi matrix $\boldsymbol{J}$. The deficiency, a corank

$$\mathrm{corank}(\boldsymbol{J}) = n - \mathrm{rank}(\boldsymbol{J}) \tag{15}$$

can serve as another factor for the classification of singularities. Typically, the $\mathrm{corank}(\boldsymbol{J}(\boldsymbol{q})) = 1$ and those singularities are unavoidable, contrary to avoidable singularities characterized by higher coranks. A corank one set forms in the n-dimensional configuration space a $(n-1)$ dimensional subspace and splits the configuration space into two pieces. If initial and final configurations of a planned motion are located not in the same component, then a continuous curve connecting them has to cross the singularity region at least once and a singular configuration is inevitably met. When a singularity is of corank two or more, then it can be avoided as the singularity set is too small to split the configuration space into separated components and a trajectory between any two configurations can be planned avoiding this set.

For a special class of manipulators with the last three motion axes crossing at a single point a kinematic decoupling technique [19] can be applied to simplify solving an inverse kinematic task. Unfortunately, even for the manipulators singularities can also be encountered.

Till now singular configurations were considered as troublemakers. In some circumstances, however, they may be even desirable. Based on the virtual work principle

$$\langle \dot{\boldsymbol{x}}, \boldsymbol{\tau} \rangle = \langle \dot{\boldsymbol{q}}, \boldsymbol{f} \rangle \tag{16}$$

one can calculate how forces/momenta $\boldsymbol{\tau}$ applied at the end-effector of a manipulator are transformed into reactions forces/momenta at joints $\boldsymbol{f}$ (cf. Fig. 1)

$$\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q})\,\boldsymbol{\tau} = \boldsymbol{f}. \tag{17}$$

In Eq. (16) $\langle \cdot\,,\ \cdot \rangle$ stands for the inner product and $\dot{\boldsymbol{q}}$, $\dot{\boldsymbol{x}}$ are velocities in the configuration/task space, respectively.

To illustrate the force/momenta transformation (17) let us, once again, analyze the double pendulum (5), (6) interpreted as a simplified model of a human standing in the up-right singular configuration, $\boldsymbol{q} = (\pi/2, 0)^{\mathrm{T}}$. In this case the gravity force $\boldsymbol{\tau} = (0, \tau_y = mg)^{\mathrm{T}}$ acts along y-axis, where m denotes a mass and g stands for the gravity acceleration $g = 9.81$ m/s^2:

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -a_1 - a_2 & 0 \\ -a_2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \tau_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{18}$$

At this particular, singular configuration no reaction forces act at joints what is the optimal pose for a human to stand for a long time. Quite different result is obtained when the optimal configuration of a simplified 2dof human hand while handwriting is searched for. This time the manipulability measure, introduced by Yoshikawa [22]

$$\mathrm{man}(\boldsymbol{q}) = \sqrt{\det(\boldsymbol{M}(\boldsymbol{q}))} \tag{19}$$

for the planar double pendulum (5) is to be optimized. After substituting (6), into (3) and (19) one obtains

$$\text{man}(\boldsymbol{q}) = a_1 a_2 \left|\sin(q_2)\right|. \tag{20}$$

Thus, the optimal configuration for handwriting is $q_2 = \pm\pi/2$ being as far as possible from singularities, cf. Eq. (7).

The aforementioned examples display a general position-force duality. Consequently, a simultaneous optimization of forces and positions is not possible and only compromised (weighted) solution is possible when both factors are important.

2.2. Nonholonomic systems

The second class of models considered result from nonholonomic constraints in the Pfaff form

$$\boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = 0. \tag{21}$$

Those constraints are due to no lateral/longitudinal slippage of wheels of mobile robots [6] or a preservation of the angular momentum for free floating robots [5].

Let us start with the unicycle robot with a single no-side slippage constraint

$$\sin(\theta)\dot{x} - \cos(\theta)\dot{y} = \boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = 0 \tag{22}$$

where the configuration $\boldsymbol{q} = (x, y, \theta)^{\mathrm{T}}$ is composed of position and orientation of the robot.

For a control purpose it is desirable to have rather a driftless control system

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u} = \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{q})u_i \tag{23}$$

than equation on constraints (21). Thus a matrix $\boldsymbol{G}(\boldsymbol{q})$ should be selected that spans a space perpendicular to rows of the matrix $\boldsymbol{A}(\boldsymbol{q})$

$$\boldsymbol{A}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{0}. \tag{24}$$

As $\dim\boldsymbol{q} = 3$ and there is a single constraint, $r = 1$ so two vector fields should be found $m = n - r = 2$ perpendicular to $\boldsymbol{A}(\boldsymbol{q})$ and independent of each other. The first is quite simple

$$\boldsymbol{g}_1(\boldsymbol{q}) = (0,0,1)^{\mathrm{T}} \tag{25}$$

but the second could be

$$\boldsymbol{g}_2(\boldsymbol{q}) = (1/\sin(\theta), 1/\cos(\theta), 0)^{\mathrm{T}} \tag{26}$$

apparently introducing a singularity at the subspace of the configuration space $(\star, \star, k\pi/2)^{\mathrm{T}}$. This type of singularity we name a modeling singularity as due to wrong model formulation (23), (25), (26), singularities were introduced. When modeling process is performed accurately, there is no modeling singularities and the truly good vector should be selected as follows

$$\boldsymbol{g}_2(\boldsymbol{q}) = (\cos(\theta), \sin(\theta), 0)^{\mathrm{T}}. \tag{27}$$

Now another family of models of the form (23) is considered

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ q_1^j \end{bmatrix} u_2 = \boldsymbol{g}_1(\boldsymbol{q})u_1 + \boldsymbol{g}_2(\boldsymbol{q})u_2, \quad j \in \mathbb{N}. \tag{28}$$

It can be checked, by direct calculations, that possible nonzero vector fields derived from generators $\boldsymbol{g}_1(\boldsymbol{q})$, $\boldsymbol{g}_2(\boldsymbol{q})$ with the use of a Lie bracket operation $[\cdot, \cdot]$ are of the form

$$\left[ad^k \boldsymbol{g}_1, \boldsymbol{g}_2\right] = \underbrace{\left[\boldsymbol{g}_1, [\ldots, [\boldsymbol{g}_1, \boldsymbol{g}_2]\ldots\right]}_{k \text{ times}}, \quad \left[ad^0 \boldsymbol{g}_1, \boldsymbol{g}_2\right] = \boldsymbol{g}_2. \tag{29}$$

In fact

$$\begin{cases} \left[ad^k \boldsymbol{g}_1, \boldsymbol{g}_2\right] = \left(0, 0, \dfrac{j!}{(j-k)!} q_1^{j-k}\right)^{\mathrm{T}}, & \text{for} \quad k \le j \\ \left[ad^k \boldsymbol{g}_1, \boldsymbol{g}_2\right] = (0,0,0)^{\mathrm{T}}, & \text{for} \quad k > j \end{cases}, \tag{30}$$

and in (30), by definition

$$q_1^0 \equiv 1, \quad \text{and} \quad 0! = 1. \tag{31}$$

From (30) it can be concluded that the system (28) is nonholonomic (and a small time locally controllable) as it satisfies the Chow theorem (its Lie algebra is of the full rank, LARC) [2]

$$\forall \boldsymbol{q} \quad \det\left[\boldsymbol{g}_1\ \boldsymbol{g}_2 \left[ad^j \boldsymbol{g}_1, \boldsymbol{g}_2\right]\right] = j! \ne 0. \tag{32}$$

It is also nilpotent, cf. Eq. (32) and singular at configurations $(0, \star, \star)^{\mathrm{T}}$ because outside the set extra one bracket

$$\forall \boldsymbol{q} \setminus (0, \star, \star)^{\mathrm{T}} \quad \det\left[\boldsymbol{g}_1\ \boldsymbol{g}_2 \left[ad^j \boldsymbol{g}_1, \boldsymbol{g}_2\right]\right] = 1 \ne 0 \tag{33}$$

is enough to satisfy LARC. What is even more important when planning a motion between configurations $(q_1(0), q_2(0), q_3(0))^{\mathrm{T}}$ and $(q_1(0), q_2(0), q_3(\mathrm{T}))^{\mathrm{T}}$ (two first coordinates are assumed to be equal as it is quite easy to get their desirable values by switching on a single control either u_1 or u_2) outside the singularity region controls in the form:

$$u_1(t) = a_1 \sin(\omega t), \qquad u_2(t) = a_2 \cos(\omega t), \tag{34}$$

are able to steer the system into the goal configuration. In Eq. (34) the base frequency $\omega = 2\pi/T$, T is a time horizon, and a_1, a_2 are variables depending on $q_3(T) - q_3(0)$. In a singularity region the controls (34) are useless as they do not change the final value of coordinate q_3 despite manipulating with amplitudes and phases of controls. It appears that only controls with appropriately increased frequency of the second control

$$u_1(t) = a_1 \sin(\omega t), \qquad u_2(t) = a_2 \cos(j\omega t), \tag{35}$$

can solve the planning problem. This example shows that at singular configurations a switching between control scenarios should be performed. Obviously one can take a mix of sinus/cosine functions but a redundant representation of controls is not desirable as it increases a computational complexity. One more message from this example is that one should know a structure of a controlled system to properly design a parameterized family of admissible controls. In this simple case one could calculate and prove which controls are good and which are useless. In a general case, with no useful information on the structure one can frequently take harmonic controls up to a given boundary frequency, but unfortunately it introduces a huge redundancy and problems with effective and fast motion planning.

In the previous example at singular configurations, cf. (32), (33), more than the minimum number of vector fields has to be used to preserve the LARC condition. In the next example at singular configurations the model cannot be used in planning at all. The kinematic car [11], belongs to the family of models given by (23) with $m = 2$ and vector fields

$$g_1(q) = (\cos\theta, \sin\theta, \tan\phi / L, 0)^T, \quad g_2 = (0,0,0,1)^T. \quad (36)$$

The configuration vector $q = (x, y, \theta, \phi)^T$ is composed of position and orientation of the mid-point of the rear axis, ϕ is the angle of the steering front wheel and L is a constant parameter, the distance between front and rear axes. The vector field $g_1(q)$ and its descendants like

$$[g_1(q), g_2] = \left(0,0,1/\left(L(\cos\phi)^2\right),0\right)^T \quad (37)$$

are badly conditioned at singular configurations characterized by $\phi = \pm\pi/2$. To avoid the singularities and to cover all cases of a practical value, the configuration space is restricted. The range of admissible angles $\phi \in (-\phi_{max}, \phi_{max})$, with $0 < \phi_{max} < \pi/2$ impacts also the minimum turning radius of the vehicle $\rho_{min} = L / \tan\phi_{max}$, [11].

A motion planning for systems (23) with an additional drift can be solved using the Endogenous Configuration Space Method [21]. The method is a variant of the Newton algorithm of inverse kinematics with kinematics specifically defined to nonholonomic systems.

The system (23) is linearized along trajectory corresponding to given controls $u(\cdot)$

$$\xi = A(t)\xi + B(t)v \quad (38)$$

with

$$A(t) = \frac{\partial G(q(t))u(t)}{\partial q}, \quad B(t) = G(q(t)) \quad (39)$$

and $v(\cdot)$ is a small variation of controls. Afterwards a Jacobi matrix is formulated based on the formula

$$J_{q_0,T}(u(\cdot))v(\cdot) = \int_0^T \Phi(T,t)B(t)v(t)\,dt \quad (40)$$

where the fundamental matrix $\Phi(T,t)$ satisfies

$$\frac{\partial\Phi(t,s)}{\partial t} = A(t)\Phi(t,s), \quad \Phi(s,s) = I_n. \quad (41)$$

In practice the matrices $\Phi(T,s)$ for $s \in [0, T]$ can not be calculated analytically. Usually, the interval $[s, T]$ is divided into K (mostly) equi-length sub-intervals

$$T - s = \Delta s \cdot K$$

and the value of $\Phi(T,s)$ is approximated as follows

$$\Phi(T,s) = \prod_{i=K}^{1}\left(I + A\left(s + (i-1/2)\Delta s\right)\Delta s\right). \quad (42)$$

where I is the identity matrix. In order to avoid too excessive computations, a recursive formula is used based on the identity:

$$\Phi(T,s) = \Phi(T,s_1)\Phi(s_1,s), \quad (43)$$

where $T > s_1 > s$. The Newton algorithm of motion planning allows to modify controls to approach end-point of a current trajectory to the desired goal configuration if only the Gramm matrix

$$GR_{q_0,T}(u(\cdot)) = \int_0^T \Phi(T,t)B(t)B^T(t)\Phi^T(T,t)\,dt \quad (44)$$

is non-singular. In this way another classification of singularities can be proposed. It appears that despite the trivial control $u(\cdot) \equiv 0$ singular configuration cannot be determined analytically as both fundamental matrix $\Phi(T,t)$ at time t, (42), and consequently the Gramm matrix GR can be calculated only numerically and one cannot be sure that the singularities do not depend on time intervals assumed while integration. For manipulators, like in the case of 2D planar pendulum, singular configurations are characterized in a pure analytic form and frequently configurations can be described with close form formulas.

It should be mentioned that the motion planning based on the endogenous configuration space method was presented in the non-parametric version. In many practical applications its parametric version is used [17] when controls $u(\cdot)$ are selected in a parametric form

$$u_i(t) = \sum_{j=1}^{N_i} \phi_j(t)\lambda_{ij}, \quad i = 1, \ldots, n, \quad (45)$$

and $\phi_j(t)$ are taken from a functional basis (for example Fourier one) on the interval $[0, T]$ and λ collects all parameters of controls. As in the non-parametric method also in this case singularities may arise and one more problem appear how to properly select the representation of controls (45).

2.3. The optimal control

A standard method in the optimal control, also used in robotic applications, is to apply the Pontriagin's Maximum Principle (PMP) [13]; [10]. For a given system of differential equations

$$\dot{q} = F(q, u) \quad (46)$$

and an integral quality function

$$J(u(\cdot)) = \int_0^T L(q(t), u(t))dt \quad (47)$$

co-state variables p are introduced. Then the Hamiltonian function $H(q, p, u)$ is formulated and Hamiltonian equations afterwards. Finally, based on the PMP equation

$$\max_{u \in \mathbb{U}} H(q^*, p^*, u) = H(q^*, p^*, u^*) \quad (48)$$

the first order necessary condition is formulated. In a typical case the condition depends explicitly on u and a control law $u^*(q^*, p^*)$ is formulated as a function of the extreme pair (q^*, p^*). After substituting the controls into the Hamiltonian equations, a two-point boundary value problem is to be solved [9]. It should be pointed out that the control law may not be determined at some (isolated) points on the time axis, not influencing substantially the resulting trajectory. Unfortunately, for some optimal control problems the first order necessary condition does not depend on u at all and some extra effort should be taken to retrieve solvability of the problem [1]. Usually, higher order optimality conditions are considered (when the first among consecutive derivatives w.r.t. u of the Hamiltonian function depends on u explicitly) or Hamiltonian function is disturbed slightly to retrieve applicability of the first order necessary optimality condition. In any case, special measures should be taken to restore the dependence of u^* on (q^*, p^*).

Singular optimal controls are more commonly encountered in economics or biology rather than in technical sciences. Such controls are illustrated on the example taken from [12], where sales optimization

$$\int_0^T (1-u)\,x\,dt \tag{49}$$

is performed for the model and the initial condition given below

$$\dot{x} = x \cdot u, \quad x(0) = x_0. \tag{50}$$

In (49), (50), $u(t)$ denotes a fraction of stock to be reinvested while $x(t)$ stock reinvested/sold. Naturally, the control is constrained $u(t) \in [0,1]$. For the fixed time $T > 1$, the optimal control takes the form

$$u^*(t) = \begin{cases} 1 & \text{for} \quad t \in [0, T-1), \\ 0 & \text{for} \quad t \in [T-1, T]. \end{cases} \tag{51}$$

and are singular on the interval $[T-1,\ T]$. In robotics bang-bang rather singular controls are encountered for the minimum-time motion planning [18], or the Hamilton function depends on a square of controls (so it is not linear) when the energy of motion is involved in the quality function.

3. Detection of Singularities

For holonomic manipulators singular configurations can be detected in two ways. A drop of the Jacobi matrix rank can be checked either by equating to zero $\binom{n}{m}$ determinants of all $(m \times m)$ square sub-matrices of the Jacobi matrix or with checking only one determinant of the manipulability matrix (3). The first method seems to be more complicated, but for $n = m + 1$ it is probably simpler that the second one, as in the case of the planar 3D-pendulum. Both methods give analytic formulas for singularities but they do not provide, easy to determine, information on a corank of the singularity. For this purpose one can use the Singular Decomposition Value, SVD, algorithm propagated in robotics by Maciejewski and Klein [14] which factorizes the Jacobi matrix into

$$\boldsymbol{J} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^{\mathrm{T}} \tag{52}$$

where $\boldsymbol{U} \in \mathbb{SO}(m)$, $\boldsymbol{V} \in \mathbb{SO}(n)$ are matrices that belong to special orthogonal groups of appropriate sizes, and $\boldsymbol{D}$ is a diagonal matrix with singular values on its main diagonal. The number of singular values with (almost) zero values is equal to the corank (15) of the Jacobi matrix at a singular configuration. Unfortunately, SVD is a purely numeric algorithm and the analytic form of the decomposition is not available.

For nonholonomic systems, the LARC is checked by generating, with the Lie brackets, more and more complex vector fields, starting with generators $\boldsymbol{g}_i(\boldsymbol{q})$ of the system (23). Then, the vector fields are added row-by-row into a matrix and the rank of the matrix is checked after each addition. Here also the checking can be simplified as instead of using all vector fields, one can take only those that belong to the Ph. Hall basis without loosing any rank information.

In many algorithms of robotics a permanent evaluation of a distance to singularities is to be performed to find a right moment to (re-)act properly. The distance depends not only on a current configuration but also on some geometrical parameters. Therefore constants used to detect a neighborhood of singularities should be somehow correlated with the parameters.

One more important issue related to a computational complexity and detection of singularities should be addressed. The detection of a neighborhood of singularities can be time-consuming, especially when performed frequently. On the other hand when a test for singularity is negative, all computations performed are useless for a regular case algorithm. Therefore a reasonable compromise between a computational effort and usefulness of results should rely on using information gathered at detecting singularities also in a regular control mode. An example for the practical and effective approach is provided when the SVD algorithm helps to compute the generalized inverse matrix of $\boldsymbol{J}$ based on the formula

$$\boldsymbol{J}^{\#} = \boldsymbol{V}\boldsymbol{D}^{-1}\boldsymbol{U}^{\mathrm{T}}, \tag{53}$$

where $\boldsymbol{D}^{-1} = \mathrm{diag}(1/d_{ii})$. Previously, the diagonal matrix $\boldsymbol{D}$ was used to detect singularities too.

4. Coping with Singularities

For holonomic systems (manipulators) the simplest method of coping with singular configurations is to make robust the badly-conditioned manipulability matrix $\boldsymbol{M}$, (3). It is done by adding to the matrix a small disturbing matrix $\xi\boldsymbol{I}_m$ and running the Newton algorithm (2). When configurations generated with the algorithm leave a neighborhood of a singularity region, the perturbation term is switched off. In practice it is advised to weight components of the identity diagonal matrix $\boldsymbol{I}_m$ with some coefficients corresponding to the importance (range and units) of coordinates of forward kinematics and make them comparable.

Another class of methods relies on the principle of extrapolation of trajectory behavior gathered before entering a singularity region into its future evolution. As an example, a tunneling method of passing through singularities proposed by Dulęba and Sąsiadek [8] can be recalled. This method extrapolates linearly singular values of the Jacobi matrix collected in matrix $\boldsymbol{D}$, cf. (52).

The more sophisticated method to deal with singularities has been developed by Tchoń and coworkers [20] being based on a normal form approach. At first it detects a type of singularity which is approached. Then, with appropriately constructed diffeomorphisms, it transforms a task from the task space into a joint space. In the latter space a trajectory is planned around singularities. The normal form method is computationally involved.

When a robot is redundant it can be tried to avoid approaching to singularities by optimization within the null space of the Jacobi matrix. In this case the right-hand side of Eq. (2) is supplemented with the following term

$$\rho\left(\boldsymbol{I}_m - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\frac{\partial f}{\partial \boldsymbol{q}}, \tag{54}$$

where a differentiable function $f(\boldsymbol{q})$ should penalize approaching singularity. A good candidate for the function is a manipulability measure (19) (without the square root that complicates the differentiation).

As it was mentioned previously, in some situations switching between a regular-case model and another well-conditioned model in vicinity of singularities is indispensable.

5. Conclusions

In this paper various exemplifications of singularities in robotics were highlighted. The singularities pose some problems in motion planning and control of robots where some algorithms are badly-conditioned while other stop to work at all. In practice a singular cases should be considered separately from regular ones. It should be also pointed out that a problem of singularities is not focused on some sub-spaces of a general space considered. It is also extended to points close to singularities when some numerical problems are encountered as well. Some methods of the detection and coping with singularities were also discussed.

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Różne oblicza osobliwości w robotyce

Streszczenie: W przeglądowym artykule przedstawiono wybrane zagadnienia dotyczące różnych koncepcji osobliwości spotykanych w robotyce. Analizowane są osobliwości w zadaniu odwrotnej kinematyki dla manipulatorów szeregowych, planowaniu ruchu układów nieholonomicznych oraz sterowaniu optymalnym. Rozważane zadania obejmują duży obszar praktycznych systemów robotycznych. Podjęto próbę zdefiniowania pojęcia osobliwości niezależne od konkretnego zadania. Zaproponowano kilka klasyfikacji osobliwości w zależności do różnych kryteriów oraz zilustrowanych na prostych przykładach. Osobliwości przeanalizowano z numerycznego i fizycznego punktu widzenia. Ogólnie, osobliwości stwarzają pewne problemy w planowaniu ruchu i/lub sterowaniu robotami. Jednakże, jak pokazano na przykładzie transformacji sił/momentów w manipulatorach szeregowych, w niektórych przypadkach mogą one być również użyteczne. Przedstawiono także techniki wykrywania osobliwości oraz metody radzenia sobie z nimi. Praca w założeniu ma charakter dydaktyczny i ma pomóc badaczom z kręgu robotyki uzyskać ogólny pogląd na zagadnienie osobliwości.

Słowa kluczowe: układy holonomiczne, układy nieholonomiczne, osobliwości, klasyfikacje, detekcja, unikanie

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