Extremal Problems for Second Order Hyperbolic Systems with Time-Varying Delays

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Abstract: Extremal problems for time-varying delay hyperbolic systems are presented. The optimal boundary control problems for hyperbolic systems in which time-varying delays appear both in the state equations and in the Neumann boundary conditions are solved. The time horizon is fixed. Making use of Dubovicki–Milutin scheme, necessary and sufficient conditions of optimality for the Neumann problem with the quadratic performance functionals and constrained control are derived.

Keywords: optimal control theory, boundary control, second order hyperbolic systems, time-varying delays, electric long lines

1. Introduction

The rapid development of manufacturing process automatization has been largely due to increasing demands with respect to the rate and quality of production, during and since the Second World War. The high production rates resulted in high speed variation of control variables. The period of time between the measurement of the controlled variable, the control decision and the control signal action therefore grew in importance. The result of a feedback action delay is critical in the case of airplanes, rockets or large remotely controlled plants. The delayed response of the control system to disturbances appearing in the process results in the closed loop system and also in the nonstability of this system. When a mathematical model incorporates time delays, equations of a more general form than that of differential equations have to be used. These are difference-differential equations which are in a class of more general functional-differential equations. A mathematical model in a difference-differential equation form includes as special cases systems described by differential equations, i.e. certain continuous-time control systems and systems described by difference equations, i.e. certain impulse control systems. Apart from the lumped delays, which lead to difference-differential equations, control systems may incorporate so-called distributed delays. These delays occur in distributed parameter systems represented by partial differential equations. Distributed delays almost always occur in long lines, e.g. electric, hydraulic or pneumatic lines. Long lines models, however can be often be well-approximated by means difference-differential equations. In control theory the notion of equivalent delay is also used. This is useful when replacing higher-order differential equations with lower-order difference-differential equations or nonlinear differential equations with linear difference-differential equations. The present theory of optimal control is an extension of the classical calculus of variations which has proved ineffective in many of the problems of modern technology. The landmark work of Pontryagin and his group, and also of Bellman in the 1950's gave birth to two new approaches to variational problems, one based on the maximum principle and the other on the optimality principle.

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Extremal problems are now playing an ever-increasing role in applications of mathematical control theory. It has been discovered that notwithstanding the great diversity of these problems, they can be approached by a unified functional-analytic approach, first suggested in 1962 by Dubovicki and Milutin. The general theory of extremal problems has been developed so intensely recently that its basic concepts may now be considered complete.

Igor V. Girsanov, was one of the first mathematicians to study general extremum problems and to realize the feasibility and desirability of a unified theory of extremal problems, based on a functional–analytic approach. His book [4] was apparently the first systematic exposition of a unified approach to the theory of extremal problems. This approach was based on the ideas of Dubovicki and Milutin concerning extremum problems in the presence of constraints.

Dubovicki and Milutin found a necessary condition for an extremum in the form of an equation set down in the language of functional analysis. The Dubovicki–Milutin method has been applied to solve many sophisticated mathematical problems. For this purpose, we shall present the following papers.

The main object of the paper [18] is to extend the Dubovicki–Milutin results and to show that the necessary optimality conditions can be derived for a broader class of optimisation problems [17, 18]. To do that, a type of conical approximation weaker than those employed by Neustadt [32, 33] and Dubovicki–Milutin [2] is introduced; and it leads to a stronger kind of separation theorem [17, 18]. Using this theorem, the necessary optimality conditions given by Neustadt [32, 33] and Dubovicki–Milutin [2] are generalised [17, 18]. The theoretical considerations are illustrated by an example.

In the paper [21], the Dubovicki–Milutin method presented in [4] is applied to obtain a necessary condition for a problem with only one equality constraint. In [37] some generalisation of the Dubovicki–Milutin method is shown for the case of n equality constraints in any form under the same assumptions about the tangent cones and the cones dual to them. This generalisation is applied in [35] and [36]. Consequently, in the paper [21] a specification of the Dubovicki–Milutin method is obtained without those assumptions from [37], but in the case of n equality constraints given in the operator form.

In the paper [22], the method of contractor direction obtained by Altman is to applied to the Dubovicki–Milutin formalism. By using this method, some specification of the Dubovicki–Milutin theorem is proved for the case of n equality constraints given by Gâteaux – differentiable operators.

Next in the paper [23] two kinds of optimal control problems of hyperbolic systems with additional equality constraints are considered: a problem with the operator equality constraint in the form of terminal condition and a problem with nonoperator equality constraint in the form $u(\cdot) \in U$, where U – some set. The extremum principles are proved: for the first problem – by using some specification of the Dubovicki–Milutin method in the case of n equality constraints in the operator form and for the second one – by using some generalisation of the Dubovicki–Milutin theory.

In the paper [24] the problem of optimal control with mixed equality and inequality operator constraints is considered under the assumption of Gâteaux differentiability. The extremum principle for this kind of problem is obtained by using some specification of the Dubovicki–Milutin method [21] for the case of n equality constrains given by Gâteaux differentiable operators.

A some specification of the Dubovicki–Milutin method is given by without any additional assumption about the cones but for the case on n equality constraints given in the operator form in [21]. This specification is applied in the paper [25] to obtain a necessary condition for the problem of an optimal control with equality constrains on the phase coordinates and the control.

In the paper [26], an optimisation problem with inequality and equality constraints in Banach spaces is considered in the case when the operators which determine the equality constraints are nonregular. In this case, the classical Euler–Lagrange equation has degenerate form, i.e., does not depend on the functional to be minimized. Applying some generalisation of the Lusternik theorem to the Dubovicki–Milutin method, the family of Euler–Lagrange equations is obtained in the nondegenerate form under the assumption of twice Fréchet differentiability of the operators. The Pareto–optimal problem is also considered.

In the paper [27], optimal control problems of systems governed by parabolic equations with an infinite number of variables and with additional equality constraints are considered. The extremum principles as well as sufficient conditions of optimality are formulated by using certain extensions of the Dubovicki–Milutin method.

In the paper [28], an optimal control problem with terminal data is considered in the so–called abnormal case, i.e., when the classical Pontryagin–type maximum principle has a degenerate form which does not depend on the functional being minimized. An extension of the Dubovicki–Milutin method to the nonregular case, obtained by applying Avakov's generalisation of the Lusternik theorem, is presented. By using this extension, a local maximum principle which has a nondegenerate form also in the abnormal case is proved. An example which supports the theory is given.

In the paper [29], a general distributed parameter control problem in Banach spaces with integral cost functional and with given initial and terminal data is considered. An extension of the Dubovicki–Milutin method to the case of nonregular operator equality constraints, based on Avakov's generalisation of the Lusternik theorem, is presented. This result is applied to obtain an extension of the Extremum Principle for the case of abnormal optimal control problems. Then a version of this problem with nonoperator equality constraints is discussed and the Extremum Principle for this problem is presented.

In the paper [30], two types of general distributed parameter control problems in the Banach space with integral cost functional, with given terminal data and with initial data, given or insufficient, are considered. By using of the Dubovicki–Milutin method and some of its extensions, the extremum principles as well as sufficient conditions of optimality are proved for both types of problems.

Combining the results of Walczak [37] and Lasiecka [17, 18], the Author [5], extended the well-known Dubovicki–Milutin theorem in two directions: (i) The Author [5] allowed for treatment of several equality constraints in optimization problems; (ii) The Author [5] assumed less regularity of optimization problems by requiring the existence of external cones instead of tangent cones. An example of an optimization problem is also considered.

In the paper [6] a distributed control problem for a parabolic operator with an infinite number of variables is considered. The performance index is more general than the quadratic one and has an integral form. Making use of the Dubovicki–Milutin theorem, necessary and sufficient conditions of optimality are derived for the Dirichlet problem.

In the paper [7] a distributed control problem with delay for the parabolic operator with an infinite number of variables is considered. The performance index has an integral form. Constraints on controls are assumed. To obtain optimality conditions, the generalization of the Dubovicki–Milutin theorem given by Walczak in [37] was applied.

In the paper [9] a distributed control problem for the parabolic operator with an infinite number of variables and time delay is considered. The performance index has an integral form. Constraints on controls are imposed. To obtain optimality conditions for the Neumann problem, the generalization of the Dubovicki–Milutin theorem given by Walczak in [36] and [37] was applied.

In the paper [8] a distributed control problem for a hyperbolic system with mixed control-state constraints involving operator of infinite order is considered. The performance index is more general than the quadratic one and has an integral form. Making use of the Dubovicki–Milutin theorem, the optimality conditions for the Neumann problem are derived.

In the paper [1] a distributed control problem for $n \times n$ parabolic coupled systems involving operators with infinite order is considered. The performance index is more general than the quadratic one and has an integral form. Constraints on controls are imposed. Making use of the Dubovicki–Milutin theorem, the necessary and sufficient conditions of optimality are derived for the Dirichlet problem. Yet, the problem considered here is more general than the problems in [34].

In the papers [13, 15] the linear quadratic problems of optimal boundary control for hyperbolic systems in which constant time lags [13] and multiple constant time lags [15] appear both in the state equations and in the Neumann boundary conditions are solved.

Sufficient conditions for the existence of a unique solution of such hyperbolic equations with the Neumann boundary conditions involving constant time delays [13] and multiple constant time lags [15] are presented. Making use of Dubovicki–Milutin method [4], necessary and sufficient conditions of optimality with the quadratic cost functions and constrained boundary control are derived for the Neumann problem.

Extremal problems for time-varying delay hyperbolic systems are investigated. The purpose of this paper is to show the use of Dubovicki–Milutin theorem [4] in solving optimal control problems for hyperbolic systems.

As an example, an optimal boundary control problem for a system described by a linear partial differential equation of hyperbolic type in which time-varying delays appear both in the state equation and in the Neumann boundary condition is considered.

Such equations constitute, in a linear approximation, a universal mathematical model for many processes in which transmission signals at a certain distance with electric, hydraulic and other long lines take place.

In the processes mentioned above time–delayed feedback signals are introduced at the boundary of a system's spatial domain. Then the signal at the boundary of a system's spatial domain at any time depends on the signal emitted earlier with various velocity. This leads to the boundary conditions involving time-varying delays.

Sufficient conditions for the existence of a unique solution of such hyperbolic equation with the Neumann boundary condition are presented.

The performance functionals have the quadratic form. The time horizon is fixed. Finally, we impose some constraints on the boundary control. Making use of the Dubovicki–Milutin theorem [4], necessary and sufficient conditions of optimality with the quadratic performance functionals and constrained control are derived for the Neumann problem.

2. Principal Notations

R	– set of real numbers.
R^n	– real n-dimensional euclidean space.
$\Omega \subset R^n$	– bounded, open set with smooth boundary Γ.
$x = \{x_1, ..., x_n\}$	– denotes the space variable such that $x \subset \Omega \subset R^n$.
t	– denotes time variable such that $t \in (0,\ T),\ T < \infty$.
$Q = \Omega \times (0,\ T)$	– cylinder.
$\Sigma = \Gamma \times (0,\ \mathrm{T})$	– boundary of the cylinder Q.
v	– boundary control.
v^0	– optimal boundary control.
$y(x,t;\ v)$	– denotes the state variable corresponding to a given control v.
$p(x,t;\ v)$	– denotes the adjoint variable corresponding to a given control v.
$I(v)$	– denotes the cost function.
$X,\ Y$	– are the Hilbert spaces.
$X',\ Y'$	– denote the dual spaces of X and Y respectively.
$A,\ B$	– denote operators.
$A^*,\ B^*$	– denote the adjoint operators to A and B respectively.
$\mathscr{D}(A)$	– domain of the operator A.
$\Vert\cdot\Vert_X$	– norm in the space X.
$< \cdot, \cdot >_X$	– denotes scalar product defined in the space X.
$\mathscr{L}(X,\ Y)$	– space of linear operators on X into Y.

3. Main function spaces

$H^m(a,b;X)$ – is a Sobolev space of order m ($m \geq 0$ and is integer) of functions defined on $(a,\ b)$ and taking value in X and such that

$$H^m(a,b;X) = \left\{ y \mid y,\, y^{(1)} = \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots, y^{(m)} = \frac{\mathrm{d}^m y}{\mathrm{d}t^m} \in L^2(a,b;X) \right\}$$

with the scalar product

$$\left\langle y, \varphi \right\rangle_{H^m(a,b;X)} = \sum_{j=0}^{n} \int_a^b \left\langle y^{(j)}(t), \varphi^{(j)}(t) \right\rangle_X \mathrm{d}t$$

$H^m(a,b;X)$ – is a Hilbert space.
$L^2(a,\ b;\ X)$ – is a space of measurable functions y defined on $(a,\ b)$ and taking values in X and such that

$$\Vert y \Vert_{L^2(a,b;X)} = \left(\int_a^b \Vert y(t) \Vert_X^2 \, \mathrm{d}t \right)^{\frac{1}{2}} < \infty$$

where: X is a Hilbert space.

Finally, for any pair or real numbers $r,\ s \geq 0$, we introduce the Sobolev space $H^{r,s}(Q)$ – a Sobolev space ([20], Vol. 2, p. 6) defined by

$$H^{r,s}(Q) = H^0\left(0,T;H^r(\Omega)\right) \cap H^s\left(0,T;H^0(\Omega)\right)$$

which is a Hilbert space normed by

$$\left(\int_0^T \Vert y(t) \Vert_{H^r(\Omega)}^2 \, \mathrm{d}t + \Vert y \Vert_{H^s(0,T;H^0(\Omega))}^2 \right)^{\frac{1}{2}}$$

where: $H^r(\Omega)$ and $H^s(0,T;H^0(\Omega))$ are defined as a interpolation of the following spaces

$$\left[H^m(\Omega), H^0(\Omega) \right]_\Theta = H^r(\Omega)$$

where: $(1 - \Theta)m = r,\ 0 < \Theta < 1$,

$$\left[H^m\left(0,T;H^0(\Omega)\right), H^0\left(0,T;H^0(\Omega)\right) \right]_\Theta = H^s\left(0,T;H^0(\Omega)\right)$$

where: $(1 - \Theta)m = s,\ 0 < \Theta < 1$.

4. Research methodology [15]

4.1. The Dubovicki–Milutin method

The Dubovicki–Milutin theorem [4] arises from the geometric form of the Hahn-Banach theorem (a theorem about the separation of convex sets). We shall show it on the example.

Let us assume that
E – a linear topological space, locally convex
$I(x)$ – a functional defined on E
$A_i,\ i = 1, 2, ..., n$ – sets in E with inner points which represent inequality constraints
B – a set in E without inner points representing equality constraint.

We must find some conditions necessary for a local minimum of the functional $I(x)$ on the set $Q = \bigcap_{i=1}^n A_i \cap B$, or find a point $x_0 \in E$, so that $I(x_0) = \min_{Q \cap U} I(x)$, where U means a certain environment of the point x_0.

We define the set $A_0 = \{x : I(x) < I(x_0)\}$.

Then we formulate the necessary condition of optimality as follows: in the environment of the local minimum point, the intersection of system of sets (the set on which the functional attains smaller values than $I(x_0)$ and the sets representing constraints) is empty or $\bigcap_{i=0}^{n} A_i \cap B = \varnothing$.

The condition $\bigcap_{i=0}^{n} A_i \cap B = \varnothing$ is also equivalent to the one in which there are approximations of the sets A_i, $i = 0, 1, ..., n$ and B instead of A_i or B ones. These approximations are cones with the vertex in a point $\{0\}$.

We shall approximate the inequality constraints by the regular admissible cones $RAC(A_i, x_0)$ $(i = 1, 2, ..., n)$, the equality constraint by the regular tangent cone $RTC(B, x_0)$ and for the performance functional we shall construct the regular improvement cone $RFC(I, x_0)$.

Then we have the necessary condition of the optimality $I(x)$ on the set $Q = \bigcap_{i=1}^{n} A_i \cap B$ in the form of Euler-Lagrange's equation $\sum_{i=0}^{n+1} f_i = 0$, where f_i $(i = 0, 1, ..., n + 1)$ are linear, continuous functionals, all of them are not equal to zero at the same time and they belong to the adjoint cones

$$f_i \in \left[RAC(A_i, x_0)\right]^*, \quad i = 1, 2, \ldots, n$$

$$f_{n+1} \in \left[RTC(B, x_0)\right]^*, \quad f_0 \in \left[RFC(I, x_0)\right]^*$$

$$\left\{f_i \in \left[RAC(A_i, x_0)\right]^* \Leftrightarrow f_i(x) \geq 0 \quad \forall x \in RAC(A_i, x_0)\right\}$$

For convex problems, i.e. problems in which the constraints are convex sets and the performance functional is convex, the Euler-Lagrange equation is the necessary and sufficient condition of optimality if only certain additional assumptions are fulfilled (the so-called Slater's condition).

4.2. Transposition method

Let us consider the following linear hyperbolic equation

$$\frac{\partial^2 y}{\partial t^2} + A(t)y = u \quad x \in \Omega, \quad t \in (0, T) \tag{1}$$

$$y(x, 0) = y_0(x) \quad x \in \Omega \tag{2}$$

$$y'(x, 0) = y_I(x) \quad x \in \Omega \tag{3}$$

$$\frac{\partial y}{\partial \eta_A}(x, t) = q \quad (x, t) \in \Gamma \times (0, T) \tag{4}$$

where $\Omega \subset R^n$ is a bounded, open set with boundary Γ, which is a C^∞-manifold of dimension $(n - 1)$. Locally, Ω is totally on one side of Γ.

$$Q = \Omega \times (0, T), \quad \bar{Q} = \bar{\Omega} \times [0, T], \quad \Sigma = \Gamma \times (0, T).$$

The operator $A(t)$ is given by

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\frac{\partial y(x, t)}{\partial x_j}\right) \tag{5}$$

and the coefficients $a_{ij}(x, t)$ are real C^∞ functions on $\bar{Q}$ (closure of Q) satisfying the following condition

$$\left.\begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(x, t)\Phi_i\Phi_j \geq v\sum_{i=1}^{n}\Phi_i^2 \quad v > 0, \quad \forall (x, t) \in \bar{Q}, \quad \forall \Phi_i \in R \\ &a_{ij} = a_{ji} \quad \forall i, j \end{aligned}\right\} \tag{6}$$

The equations (1)–(4) constitute a Neumann problem. The left hand side of (4) is written in the following form.

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ij}(x, t)\cos(n, x_i)\frac{\partial y(x, t)}{\partial x_j} = q(x, t) \tag{7}$$

where $\dfrac{\partial y}{\partial \eta_A}$ is a derivative at Γ, directed towards the exterior of Ω, $\cos(n, x_i)$ is i-th direction cosine of n and n being the normal at Γ exterior to Ω.

Let us consider linear hyperbolic equations (1)–(4)

We shall prove the existence of a unique solution of the mixed initial-boundary value problem (1)–(4) defined by transposition ([20]: Vol. 2, pp. 105–108 and 130–133), i.e.

$$\left\langle y, \omega'' + A\omega\right\rangle = L(\omega) \quad \forall \omega \in X^1(Q) \tag{8}$$

where

$$L(\omega) = \left\langle u, \omega\right\rangle + \left\langle q, \omega\right\rangle + \left\langle y_I, \omega(0)\right\rangle - \left\langle y_0, \omega'(0)\right\rangle \tag{9}$$

and we denote by $X^1(Q)$ the space described by the solutions ω of the following adjoint problem

$$\left.\begin{aligned} &\omega'' + A\omega = \Phi \quad x \in \Omega, \quad t \in (0, T) \\ &\omega(x, T) = 0 \quad x \in \Omega \\ &\omega'(x, T) = 0 \quad x \in \Omega \\ &\frac{\partial \omega}{\partial \eta_A} = 0 \quad x \in \Gamma, \quad t \in (0, T) \end{aligned}\right\} \tag{10}$$

where

$$\Phi \in H_{0,0}^{1,2}(Q) = \text{closure of } D(Q) \text{ in } H^{1,2}(Q). \tag{11}$$

Some properties and central theorems for the spaces $D(Q)$ and $H^{r,s}(Q)$ are given in [11] and [20].

Let us define by

$$X^1(Q) = \text{space described by } \omega \text{ as } \Phi \text{ describes } H_{0,0}^{1,2}(Q) \tag{12}$$

and then ($X^1(Q)$ being provided with the "translated" topology) we have (done what was necessary to have):

$$\omega \to \omega'' + A\omega \text{ is an isomorphism of } X^1(Q) \to H_{0,0}^{1,2}(Q) \tag{13}$$

Consequently, using the results of Chapter 4, Section 8.3 of ([20]: Vol. 2, p. 41), we may define the dual space of $H_{0,0}^{1,2}(Q)$. We denote this dual space by $H^{-1,-2}(Q)$, i.e. we set

$$\left(H_{0,0}^{1,2}(Q)\right)' = H^{-1,-2}(Q) \tag{14}$$

Now, we choose in (9) u, q, y_i to be (suitable) distributions on Q, Σ and Ω.

Thus, we shall follow a procedure analogous to Chapter 5, Section 10, of ([20]: Vol. 2, pp. 130–132). Moreover, Q and Σ have the same properties as in the problem (1)–(4).

4.2.1. Choice of *u*

According to Theorem 7.1 of ([20]: Vol. 2, p. 122), we have

$$X^1(Q) \subset H^{3,3}(Q) \tag{15}$$

Subsequently, we introduce as in Chapter 4, Section 9, of ([20]: Vol. 2, p. 43) the space

$$\Xi^{3,3}(Q) = \left\{\omega \mid d^j(t)\omega^{(j)} \in L^2\left(0,T;\Xi^{3-j}(\Omega)\right), \quad 0 \le j \le 3\right\} \tag{16}$$

where $d(t)$ be a fixed infinitely differentiable function on $[0, T]$ such that

$$d(t) = \begin{cases} t & \text{if } t \le t_0 \\ T-t & \text{if } T-t_0 \le t \le T \end{cases} \tag{17}$$

t_0 fixed with $0 < t_0 < T - t_0$, and consequently, using the results of ([20]: Vol. 1, pp. 170–173) we may define the following space

$$\Xi^{\mu}(\Omega) = \left\{\omega \mid \rho^{|\alpha|}D^{\alpha}\omega \in L^2(\Omega), \quad |\alpha| \le \mu\right\} \tag{18}$$

which is a Hilbert space with the norm

$$\|\omega\|_{\Xi^{\mu}(\Omega)} = \left(\sum_{|\alpha|\le\mu}\left\|\rho^{|\alpha|}D^{\alpha}\omega\right\|^2_{L^2(\Omega)}\right)^{1/2} \tag{19}$$

where integer $\mu \ge 1$, and the function ρ is infinitely differentiable on $\bar{\Omega}$, positive on Ω, vanishing on Γ of the order of $d(x, \Gamma)$ (= distance from x to Γ), i.e. such that

$$\lim_{x\to x_0}\frac{\rho(x)}{d(x,\Gamma)} = d \ne 0 \quad \text{if} \quad x_0 \in \Gamma, \tag{20}$$

such functions do exist, since Γ is an infinitely differentiable variety. Moreover,

$$D^{\alpha} = \frac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}+\ldots+\partial x_n^{\alpha_n}}, \quad \alpha = \{\alpha_1,\ldots,\alpha_n\}, \quad |\alpha| = \alpha_1+\ldots+\alpha_n \tag{21}$$

Moreover, we have

$$\Xi^0(\Omega) = L^2(\Omega), \quad H^{\mu}(\Omega) \subset \Xi^{\mu}(\Omega) \subset L^2(\Omega) \tag{22}$$

Then, using the Proposition 9.2 of ([20]: Vol. 2, p. 45), we have

$$H^{3,3}(Q) \subset \Xi^{3,3}(Q) \tag{23}$$

Remark 1 *In the case when μ is a non-integer, we define the space $\Xi^{\mu}(\Omega)$ by interpolation. Consequently, let real $\mu > 0$ not be an integer, $\mu = k + \Theta$, with integer $k \ge 0$ and $0 < \Theta < 1$; we set*

$$\Xi^{\mu}(\Omega) = \left[\Xi^{k+1}(\Omega), \Xi^{k}(\Omega)\right]_{1-\Theta} \tag{24}$$

From this definition, (22) and the inclusion properties of $H^{\mu}(\Omega)$, there results that

$$H^{\mu}(\Omega) \subset \Xi^{\mu}(\Omega) \subset \Xi^{\mu'}(\Omega) \subset L^2(\Omega) \tag{25}$$

where μ, μ' are real and > 0, $\mu' < \mu$.

From the density theorem and Proposition 6.2 of ([20]: Vol. 1, p. 11 and p. 171), we also deduce that $D(\Omega)$ is dense in $\Xi^{\mu}(\Omega)$ for all real $\mu \ge 0$.
Therefore, we see that $\Xi^{\mu}(\Omega)$ is a normal space of distributions on Ω.

Its dual space may be identified to a space of distributions on Ω. We denote this dual space by $\Xi^{-\mu}(\Omega)$, i.e. we set

$$\Xi^{-\mu}(\Omega) = \left(\Xi^{\mu}(\Omega)\right)', \quad \mu > 0 \tag{26}$$

As for Proposition 9.1 of Chapter 4 of ([20]: Vol. 2, p. 43), we verify that

$$D(Q) \text{ is dense in } \Xi^{3,3}(Q) \tag{27}$$

Therefore

$$\Xi^{-3,-3}(Q) = \left(\Xi^{3,3}(Q)\right)' \subset D'(Q) \tag{28}$$

For this space, we have a structure result analogous to Proposition 9.3 given in Chapter 4 of ([20]: Vol.2, p. 45) and, thanks to (15), we have:

$$\text{if } u \in \Xi^{-3,-3}(Q), \text{ then } \omega \to \left\langle u,\omega\right\rangle \tag{29}$$

is continuous antilinear on $X^1(Q)$.

4.2.2. The space $D^{-1}_{A+D_t^2}(Q)$

Taking, in (8), $\omega = \Phi \in D(Q)$, we have $L(\Phi) = \left\langle f,\Phi\right\rangle_Q$ and therefore, in the sense of $D'(Q)$:

$$y'' + Ay = u \tag{30}$$

This leads to the following definition ([20]: Vol. 2, p. 131):

$$D^{-1}_{A+D_t^2}(Q) \overset{df}{=} \left\{y \mid y \in H^{-1,-2}(Q), \quad y'' + Ay \in \Xi^{-3,-3}(Q)\right\} \tag{31}$$

provided with the norm of the graph, this is a Hilbert space.

Then the solution y of (8) belongs to $D^{-1}_{A+D_t^2}(Q)$.

4.2.3. Choice of q

Consequently, we choose q.
From (15) and the Trace Theorem ([11] and [20]: Vol. 2, p. 9), we deduce that $\omega \to \omega|_{\Sigma}$ is a continuous linear mapping of

$$X^1(Q) \to H^{5/2,5/2}(\Sigma) \tag{32}$$

Consequently, we introduce as in Chapter 4, Section 11 of ([20]: Vol. 2, pp. 57–59) the spaces $H^{\alpha}\Xi^{\alpha}(\Sigma)$, real $\alpha \ge 0$;
first for integer:

$$H^{\alpha}\Xi^{\alpha}(\Sigma) = \left\{\omega \mid d^j(t)\omega^{(j)} \in L^2\left(0,T;H^{\alpha-j/\alpha}(\Gamma)\right), \quad 0 \le j \le \alpha\right\} \tag{33}$$

where the function $d(t)$ is defined in (17), and then for non-integer $\alpha \in R_+$, we define by interpolation

$$\left.\begin{aligned} &H^{\alpha}\Xi^{\alpha}(\Sigma) = \left[H^{\alpha_0}\Xi^{\alpha_0}(\Sigma), H^{0,0}(\Sigma)\right]_{\Theta} \\ &\text{integer } \alpha_0, \quad (1-\Theta)\alpha_0 = \alpha \end{aligned}\right\} \tag{34}$$

The space defined in this way depends only on α. Then:

$$H^{5/2,5/2}(\Sigma) \subset H^{5/2}\Xi^{5/2}(\Sigma) \tag{35}$$

As for Proposition 11.1, Chapter 4 of ([20]: Vol. 2, p. 58), we verify that

$$D(\Sigma) \text{ is dense in } H^{\alpha}\Xi^{\alpha}(\Sigma) \tag{36}$$

and therefore

$$H^{-\alpha}\Xi^{-\alpha}(\Sigma) = \left(H^{\alpha}\Xi^{\alpha}(\Sigma)\right)' \subset D'(\Sigma) \tag{37}$$

According to (35), we have:

$$\left.\begin{array}{l}\text{if } q \in H^{-5/2}\Xi^{-5/2}(\Sigma), \text{ then } \omega \to \left\langle q,\omega\right\rangle \\ \text{is continuous on } X^1(Q)\end{array}\right\} \tag{38}$$

4.2.4. Choice of y_0 and y_I

Subsequently, we make choice of y_0 and y_I respectively. Then, from the Theorem 2.1 of ([20]: Vol. 2, p. 9), it follows that $\omega \to \{\omega(0), \omega'(0)\}$ is a continuous mapping of

$$H^{3,3}(Q) \to H^{5/2}(\Omega) \times H^{3/2}(\Omega) \tag{39}$$

Using the spaces $\Xi^{\mu}(\Omega)$, $\Xi^{-\mu}(\Omega)$ defined in (24) and (26), it follows that:

$$\left.\begin{array}{l}\text{if } \{y_0, y_I\} \in \Xi^{-3/2}(\Omega) \times \Xi^{-5/2}(\Omega), \\ \text{then the form } \omega \to \left\langle y_I, \omega(0)\right\rangle - \left\langle y_0, \omega'(0)\right\rangle \\ \text{is continuous on } X^1(Q)\end{array}\right\} \tag{40}$$

4.2.5. Central result

Consequently, the preceding results may be summarized by

Theorem 1 *Let* y_0, y_I, q *and* u *be given, with* $y_0 \in \Xi^{-3/2}(\Omega)$, $y_I \in \Xi^{-5/2}(\Omega)$, $q \in H^{-5/2}\Xi^{-5/2}(\Sigma)$ *and* $u \in \Xi^{-3,-3}(Q)$. *Then, there exists a unique solution* $y \in D^{-1}_{A+D_t^2}(Q)$ *for the problem* (1)–(4) *defined by transposition* (8).

This theorem will be the starting point for our considerations.

5. Analysis of time-varying delay hyperbolic systems

5.1. Existence and uniqueness of solutions: $v \in L^2(\Sigma)$

Consider now the distributed-parameter system described by the following hyperbolic delay equation

$$\frac{\partial^2 y}{\partial t^2} + A(t)y + y(x, t - h(t)) = u \quad x \in \Omega, \quad t \in (0,T) \tag{41}$$

$$y(x,t') = \Phi_0(x,t') \quad x \in \Omega, \quad t' \in [-h(0), 0) \tag{42}$$

$$y(x,0) = y_0(x) \quad x \in \Omega \tag{43}$$

$$y'(x,0) = y_I(x) \quad x \in \Omega \tag{44}$$

$$\frac{\partial y}{\partial \eta_A} = y(x, t - h(t)) + Gv \quad x \in \Gamma, \quad t \in (0,T) \tag{45}$$

$$y(x,t') = \Psi_0(x,t') \quad x \in \Gamma, \quad t' \in [-h(0), 0) \tag{46}$$

where Ω has the same properties as in the problem (1)–(4), $h(t)$ is a function representing time-varying delay, Φ_0, Ψ_0 are initial functions defined on Q_0 and Σ_0 respectively.

The operator $A(t)$ has the form given by (5) and (6).

$$y \equiv y(x,t;v), \quad u \equiv u(x,t), \quad v \equiv v(x,t),$$

$$Q = \Omega \times (0,T), \quad \bar{Q} = \bar{\Omega} \times [0,T], \quad Q_0 = \Omega \times \left[-h(0), 0\right)$$

$$\Sigma = \Gamma \times (0,T), \quad \Sigma_0 = \Gamma \times \left[-h(0), 0\right)$$

G is a linear continuous operator on $L^2(\Sigma)$ into $\left(H^{5/2}\Xi^{5/2}(\Sigma)\right)'$ with $v \in L^2(\Sigma)$ and $Gv \in H^{-5/2}\Xi^{-5/2}(\Sigma)$.

The operator $A(t)$ is given by the formulas (5) and(6).

It is easy to notice that the equations (41)–(46) constitute the Neumann problem. The left-hand side of the Neumann boundary condition (45) is written in the following form

$$\frac{\partial y}{\partial \eta_A} = q(x,t) \quad x \in \Gamma, \quad t \in (0,T) \tag{47}$$

where

$$q(x,t) = y(x, t - h(t)) + Gv(x,t) \quad x \in \Gamma, \quad t \in (0,T) \tag{48}$$

Let $t - h(t)$ be a strictly increasing function, $h(t)$ being non-negative in $[0, T]$ and also being a C^1 function. Then the inverse function of $t - h(t)$ exists.

Let us write $r(t) \overset{df}{=} t - h(t)$, then the inverse function of $r(t)$ has the form $t = f(r) = r + s(r)$, where $s(r)$ is a time-varying prediction. Let $f(t)$ be the inverse function of $t - h(t)$.

Thus we shall define the following iteration

$$f_j(t) \overset{df}{=} \underbrace{f \ldots \left[f[f(t)] \right]}_{j}$$

such that $f_0(t) = t$ where $f_j(t)$ is a j-th iteration of the operation $f(t)$ for $j = 0, 1, \ldots$.

We shall prove the existence of a unique solution of the mixed initial-boundary value problem (41)–(46) defined by transposition, i.e.

$$\left\langle y, \omega'' + A\omega \right\rangle = L(\omega) \quad \forall \omega \in X^1(Q) \tag{49}$$

where

$$L(\omega) = \left\langle l, \omega \right\rangle + \left\langle q, \omega \right\rangle + \left\langle y_I, \omega(0) \right\rangle - \left\langle y_0, \omega'(0) \right\rangle \tag{50}$$

and

$$L = \left[u - y(x, t - h(t)) \right] |_{Q} \tag{51}$$

and $X^1(Q)$ is the space described by the solutions ω of the adjoint problem (10).

We shall restrict our considerations to the case where $v \in L^2(\Sigma)$. For simplicity, we shall introduce the following notations

$$I_j \overset{df}{=} \left(f_{j-1}(0) f_j(0)\right), \quad Q_j = \Omega \times I_j, \quad \Sigma_j = \Gamma \times I_j \quad \text{for} \quad j = 1, 2, \ldots$$

The existence of a unique solution for the mixed initial-boundary value problem (41)–(46) on the cylinder Q can be proved using a constructive method, i.e. first, solving problem (49) in the subcylinder Q_1 and in turn in Q_2 etc. until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

Using the Theorem 1 we can prove the following lemma.

Lemma 1 *Let*

$$u \in \Xi^{-3,-3}(Q) \tag{52}$$

$$l_j \in \Xi^{-3,-3}(Q_j) \tag{53}$$

where

$$l_j = \left[u - y_{j-1}(x, t - h(t)) \right] |_{Q_j}$$

$$q_j \in H^{-5/2}\Xi^{-5/2}(\Sigma_j) \tag{54}$$

where

$$q_j = \left[y_{j-1}(x, t - h(t)) + Gv(x, t) \right] |_{\Sigma_j}$$

and

$$G \in \alpha\left(\mathcal{L}^2(\Sigma), H^{-5/2}\Xi^{-5/2}(\Sigma_j)\right) \quad \text{with} \quad v \in L^2(\Sigma)$$

$$y_{j-1}\left(\cdot, f_{j-1}(0)\right) \in \Xi^{-3/2}(\Omega) \tag{55}$$

$$y'_{j-1}\left(\cdot, f_{j-1}(0)\right) \in \Xi^{-5/2}(\Omega) \tag{56}$$

Then, there exists a unique solution $y_j \in D^{-1}_{A+D_t^2}(Q_j)$ *for the mixed initial-boundary value problem* (41), (45), (55), (56) *defined by transposition, i.e.*

$$\left\langle y_j, \omega_j'' + A\omega_j \right\rangle = L(\omega_j) \quad \forall \omega_j \in X^1(Q_j) \tag{57}$$

where

$$L(\omega_j) = \left\langle l_j, \omega_j \right\rangle + \left\langle q_j, \omega_j \right\rangle + \\ + \left\langle y'_{j-1}[f_{j-1}(0)], \, \omega_j[f_{j-1}(0)] \right\rangle - \\ - \left\langle y_{j-1}[f_{j-1}(0)], \, \omega_j'[f_{j-1}(0)] \right\rangle.$$

Outline of the proof:

For $j = 1$, conditions (53)–(56) can be satisfied if we assume that

$$\Phi_0 \in H^{-1,-2}(Q_0), \quad \Psi_0 \in H^{-5/2}\Xi^{-5/2}(\Sigma_0)$$

and

$$Gv \in H^{-5/2}\Xi^{-5/2}(\Sigma), \quad y_0 \in \Xi^{-3/2}(\Omega), \quad y_1 \in \Xi^{-5/2}(\Omega).$$

These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in D^{-1}_{A+D_t^2}(Q_1) \subset H^{-1,-2}(Q_1)$. In order to extend the result to Q_j, $1 < j \le K$ it is sufficient to verify that

$$l_2 \in \Xi^{-3,-3}(Q_2) \tag{58}$$

$$q_2 \in H^{-5/2}\Xi^{-5/2}(\Sigma_2) \tag{59}$$

$$y_1\left(\cdot, f_1(0)\right) \in \Xi^{-3/2}(\Omega) \tag{60}$$

$$y_1'\left(\cdot, f_1(0)\right) \in \Xi^{-5/2}(\Omega) \tag{61}$$

According to (15) and (23) of Subsubsection 3.2.1 we have $X^1(Q_2) \subset H^{3,3}(Q_2)$ and obviously $H^{3,3}(Q_2) \subset \Xi^{3,3}(Q_2)$. Using the results of Subsubsection 3.2.1, we have: if $y_1 \in H^{-1,-2}(Q_1)$ and $l_2 \in \Xi^{-3,-3}(Q_2)$, then $\omega_2 \to \left\langle l_2, \omega_2 \right\rangle$ is continuous antilinear on $X^1(Q_2)$ and the condition (58) is fulfilled.

To verify (59) we use the results of Subsubsection 3.2.3. Then, from the inclusion $X^1(Q_2) \subset H^{3,3}(Q_2)$, and the Theorem 2.1 ([20]: Vol. 2, p. 9) we deduce that $\omega_2 \to \omega_2 |_{\Sigma_2}$ is a linear continuous mapping of $X^1(Q_2) \to H^{5/2,5/2}(\Sigma_2) \subset H^{5/2}\Xi^{5/2}(\Sigma_2)$. According to the fact mentioned above we have if $y_1 |_{\Sigma_1} \in H^{-5/2}\Xi^{-5/2}(\Sigma_1)$ and $q_2 \in H^{-5/2}\Xi^{-5/2}(\Sigma_2)$, then $\omega_2 \to \left\langle q_2, \omega_2 \right\rangle$ is continuous on $X^1(Q_2)$.

Consequently, using the results of the Subsubsection 3.2.4, we shall verify the conditions (60) and (61). From the Theorems 3.1 and 9.6 ([20]: Vol. 1, pp. 19 and 43) $\omega_2 \in X^1(Q_2) \subset H^{3,3}(Q_2)$ implies that the mappings $t \to \omega_2(\cdot, t)$ and $t \to \omega_2'(\cdot, t)$ are continuous from $\left[f_1(0), f_2(0)\right] \to H^{5/2}(\Omega)$ and $\left[f_1(0), f_2(0)\right] \to H^{3/2}(\Omega)$ respectively. Hence $\omega_2\left(\cdot, f_1(0)\right) \in H^{5/2}(\Omega)$ and $\omega_2'\left(\cdot, f_1(0)\right) \in H^{5/2}(\Omega)$. Then, from (39) of Subsubsection 3.2.4, it follows that $\omega_2 \to \left(\omega_2(f_1(0)), \omega_2'(f_1(0))\right)$ is a continuous mapping of $H^{3,3}(Q_2) \to H^{5/2}(\Omega) \times H^{3/2}(\Omega)$.

Using the spaces $\Xi^{\alpha}(\Omega)$, $\Xi^{-\alpha}(\Omega)$ defined in (24) and (26) we deduce that if $\left(y_1(f_1(0)), y_1'(f_1(0))\right) \in \Xi^{-3/2}(\Omega) \times \Xi^{-5/2}(\Omega)$, then the form $\omega_2 \to \left\langle y_1'(f_1(0)), \omega_2(f_1(0)) \right\rangle - \left\langle y_1(f_1(0)), \omega_2'(f_1(0)) \right\rangle$ is continuous on $X^1(Q_2)$. The conditions (60) and (61) are fulfilled. Then, there exists a unique solution $y_2 \in D^{-1}_{A+D_t^2}(Q_2)$.

The foregoing result is now summarized.

Theorem 2 *Let* y_0, y_I, Φ_0, Ψ_0, v, *and* u *be given with*

$$y_0 \in \Xi^{-3/2}(\Omega), \; y_I \in \Xi^{-5/2}(\Omega), \; \Phi_0 \in H^{-1,-2}(Q_0),$$

$$\Psi_0 \in H^{-5/2}\Xi^{-5/2}(\Sigma_0), \; v \in L^2(\Sigma), \; \textit{and} \; u \in \Xi^{-3,-3}(Q).$$

Then, there exists a unique solution $y \in D^{-1}_{A+D_t^2}(Q)$ *for the problem* (41)–(46) *defined by transposition* (49). *Moreover,* $y\left(\cdot, f_j(0)\right) \in \Xi^{-3/2}(\Omega)$ *and* $y'\left(\cdot, f_j(0)\right) \in \Xi^{-5/2}(\Omega)$ *for* $j = 1, \ldots$

5.2. Problem formulation. Optimization theorems

We shall now formulate the optimal boundary control problem in the context of the case where $v \in L^2(\Sigma)$. Let us denote by $U \in L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem. The performance functional is given by

$$I(v) = \lambda_1 \left\| y(v) - z_d \right\|^2_{H^{-1,-2}(Q)} + \lambda_2 \left\langle Nv,\, v \right\rangle_{L^2(\Sigma)} \tag{62}$$

where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 > 0$; z_d is a given element in $H^{-1,-2}(Q)$, and N is a positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$. Finally, we assume the following constraint on controls

$$v \in U_{ad} \tag{63}$$

where U_{ad} is a closed, convex set with non-empty interior, a subset of U.

Let $y(x,\, t,\, v)$ denote the solution of (41)–(46), (62), (63) at $(x,\, t)$ corresponding to a given control $v \in U_{ad}$. We note from the Theorem 2 that for any $v \in U_{ad}$ the cost function (62) is well defined since $y \in D^{-1}_{A+D_t^2}(Q) \subset H^{-1,-2}(Q)$.

The optimal control problem (41)–(46), (62), (63) will be solved as the optimization one in which the function v is the unknown function. Making use of Dubovicki–Milutin theorem [4] we shall derive the necessary and sufficient conditions of optimality for the optimization problem (41)–(46), (62), (63). The solution of the stated optimal control problem is equivalent to seeking a pair $(y^0, v^0) \in E = D^{-1}_{A+D_t^2}(Q) \times L^2(\Sigma)$ which satisfies the equation (41)–(46) and minimizing the performance functional (62) with the constraints on boundary control (63).

We formulate the necessary and sufficient conditions of the optimality in the form of Theorem 3.

Theorem 3. *The solution of the optimization problem* (41)–(46), (62), (63) *exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities.*

State equations:

$$\frac{\partial^2 y^0}{\partial t^2} + A(t)y^0 + y^0\left(x, t - h(t)\right) = f \quad x \in \Omega, \quad t \in (0, T) \tag{64}$$

$$y^0(x,t') = \Phi_0(x,t') \quad x \in \Omega, \quad t' \in \left[-h(0), 0\right) \tag{65}$$

$$y^0(x,0) = y_1(x) \quad x \in \Omega \tag{66}$$

$$\frac{\partial y^0(x,0)}{\partial t} = y_2(x) \quad x \in \Omega \tag{67}$$

$$\frac{\partial y^0}{\partial \eta_A} = y^0\left(x, t - h(t)\right) + Gv^0 \quad x \in \Gamma, \quad t \in (0, T) \tag{68}$$

$$y^0(x,t') = \Psi_0(x,t') \quad x \in \Gamma, \quad t' \in \left[-h(0), 0\right) \tag{69}$$

Adjoint equations

$$\frac{\partial^2 p}{\partial t^2} + A(t)p + p\left(x, t + s(t)\right)\left(1 + s'(t)\right) = \lambda_1 \Lambda_1 (y^0 - z_d)$$
$$x \in \Omega, \quad t \in \left(0, T - h(T)\right) \tag{70}$$

$$\frac{\partial^2 p}{\partial t^2} + A(t)p = \lambda_1 \Lambda_1 (y^0 - z_d) \quad x \in \Omega, \quad t \in \left(T - h(T), T\right) \tag{71}$$

$$\frac{\partial p}{\partial \eta_A} = p\left(x, t + h(t)\right)\left(1 + s'(t)\right) \quad x \in \Gamma, \quad t \in \left(0, T - h(T)\right) \tag{72}$$

$$\frac{\partial p}{\partial \eta_A} = 0 \quad x \in \Gamma, \quad t \in \left(T - h(T), T\right) \tag{73}$$

$$p(x,T) = 0 \quad x \in \Omega \tag{74}$$

$$\frac{\partial p(x,T)}{\partial t} = 0 \quad x \in \Omega \tag{75}$$

Maximum condition

$$\left\langle G^* p(v^0) + \lambda_2 N v^0,\, v - v^0 \right\rangle_{L^2(\Sigma)} \geq 0 \quad \forall v \in U_{ad} \tag{76}$$

We can also notice that

$$\frac{\partial p}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ji}(x,t) \cos(n, x_i) \frac{\partial p}{\partial x_j} \tag{77}$$

Outline of the proof:

According to the Dubovicki–Milutin theorem [4], we approximate the set representing the inequality constraints by the regular admissible cone, the equality constraints by the regular tangent cone and the performance functional by the regular improvement cone.

a) Analysis of the equality constraint

The set Q_1 representing the equality constraint has the form

$$Q_1 = \left\{ \begin{array}{lll} \dfrac{\partial^2 y}{\partial t^2} + A(t)y + y\left(x, t - h(t)\right) = u & x \in \Omega, & t \in (0,T) \\ y(x,t') = \Phi_0(x,t') & x \in \Omega, & t' \in \left[-h(0), 0\right) \\ y(x,0) = y_1(x) & x \in \Omega & \\ \dfrac{\partial y(x,0)}{\partial t} = y_2(x) & x \in \Omega & \\ \dfrac{\partial y}{\partial \eta_A} = y\left(x, t - h(t)\right) + Gv & x \in \Gamma, & t \in (0,T) \\ y(x,t') = \Psi_0(x,t') & x \in \Gamma, & t' \in \left[-h(0), 0\right) \end{array} \right\} \tag{78}$$

We construct the regular tangent cone of the set Q_1 using the Lusternik theorem (Theorem 9.1 [4]). For this purpose, we define the operator P in the form

$$P(y,v) = \left(\frac{\partial^2 y}{\partial t^2} + Ay + y\ \left(x, t - h(t)\right) - u, \right.$$
$$y(x,t') - \Phi_0(x,t'), \quad y(x,0) - y_1(x), \quad \frac{\partial y(x,0)}{\partial t} - y_2(x),$$
$$\left. \frac{\partial y}{\partial \eta_A} - y\left(x, t - h(t)\right) - Gv, \quad y(x,t') - \Psi_0(x,t') \right) \tag{79}$$

The operator P is the mapping from the space $D^{-1}_{A+D_t^2}(Q)\times L^2(\Sigma)$ into the space

$$\Xi^{-3,-3}(Q)\times D^{-1}_{A+D_t^2}(Q_0)\times\Xi^{-3/2}(\Omega)\times\Xi^{-5/2}(\Omega)\times$$
$$\times H^{-5/2}\Xi^{-5/2}(\Sigma)\times H^{-5/2}\Xi^{-5/2}(\Sigma_0).$$

The Fréchet differential of the operator P can be written in the following form:

$$P'(y^0,v^0)(\overline{y},\overline{v})=\left(\frac{\partial^2\overline{y}}{\partial t^2}+A\overline{y}+\overline{y}\left(x,t-h(t)\right),\quad \overline{y}\,|_{Q_0}\;(x,t'),\right.$$
$$\left.\overline{y}(x,0),\quad \frac{\partial\overline{y}(x,0)}{\partial t},\quad \frac{\partial\overline{y}}{\partial\eta_A}-\overline{y}\left(x,t-h(t)\right)-G\overline{v},\quad \overline{y}\,|_{\Sigma_0}\;(x,t')\right) \tag{80}$$

Really, $\partial^2/\partial t^2$ (Theorem 2.8 [31]), $A(t)$ (Theorem 2.1 [19]) and $\partial/\partial\eta_A$ (Theorem 2.1 [20]; Vol. 2, p. 9) are linear and bounded mappings.

Using Theorem 2, we can prove that P' is the operator "one to one" from the space $D^{-1}_{A+D_t^2}(Q)\times L^2(\Sigma)$ onto the space

$$\Xi^{-3,-3}(Q)\times D^{-1}_{A+D_t^2}(Q_0)\times\Xi^{-3/2}(\Omega)\times\Xi^{-5/2}(\Omega)\times$$
$$\times H^{-5/2}\Xi^{-5/2}(\Sigma)\times H^{-5/2}\Xi^{-5/2}(\Sigma_0).$$

Considering that the assumptions of the Lusternik's theorem are fulfilled, we can write down the regular tangent cone for the set Q_1 in a point (y^0, v^0) in the form

$$RTC\left(Q_1,(y^0,v^0)\right)=\left\{(\overline{y},\overline{v})\in E,\quad P'(y^0,v^0)(\overline{y},\overline{v})=0\right\} \tag{81}$$

It is easy to notice that it is a subspace. Therefore, using Theorem 10.1 [4] we know the form of the functional belonging to the adjoint cone

$$f_1(\overline{y},\overline{v})=0\qquad\forall(\overline{y},\overline{v})\in RTC\left(Q_1,(y^0,v^0)\right) \tag{82}$$

b) Analysis of the constraint on controls

The set $Q_2=Y\times U_{ad}$ representing the inequality constraints is a closed and convex one with non-empty interior in the space E.

Using Theorem 10.5 [4] we find the functional belonging to the adjoint regular admissible cone, i.e.

$$f_2(\overline{y},\overline{v})\in\left[RAC\left(Q_2,(y^0,v^0)\right)\right]^*$$

We can note if E_1, E_2 are two linear topological spaces, then the adjoint space to $E=E_1\times E_2$ has the form

$$E^*=\left\{f=(f_1,f_2);\quad f_1\in E_1^*,\quad f_2\in E_2^*\right\}$$

and

$$f(x)=f_1(x_1)+f_2(x_2)$$

So we note the functional $f_2(\overline{y},\overline{v})$ as follows

$$f_2(\overline{y},\overline{v})=f_1'(\overline{y})+f_2'(\overline{v}) \tag{83}$$

where: $f_1'(\overline{y})=0\quad\forall y\in Y$ (Theorem 10.1 [4]); $f_2'(\overline{v})$ is a support functional to the set U_{ad} in a point v_0 (Theorem 10.5 [4]).

c) Analysis of the performance functional

Using Theorem 7.5 [4] we find the regular improvement cone of the performance functional (62)

$$RFC\left(I,(y^0,v^0)\right)=\left\{(\overline{y},\overline{v})\in E,\quad I'(y^0,v^0)(\overline{y},\overline{v})<0\right\} \tag{84}$$

where $I'(y^0,v^0)(\overline{y},\overline{v})$ is the Fréchet differential of the performance functional (62) and it can be written as

$$I'(y^0,v^0)(\overline{y},\overline{v})=2\lambda_0\lambda_1\left\langle\Lambda_1(y^0-z_d),\,\overline{y}\right\rangle_{H^{-1,-2}(Q)}+2\lambda_0\lambda_2\left\langle Nv^0,\,\overline{v}\right\rangle_{L^2(\Sigma)} \tag{85}$$

On the basis of Theorem 10.2 [4] we find the functional belonging to the adjoint regular improvement cone, which has the form

$$f_3(\overline{y},\overline{v})=-\lambda_0\lambda_1\left\langle\Lambda_1(y^0-z_d),\,\overline{y}\right\rangle_{H^{-1,-2}(Q)}-\lambda_0\lambda_2\left\langle Nv^0,\,\overline{v}\right\rangle_{L^2(\Sigma)} \tag{86}$$

where $\lambda_0>0$.

d) Analysis of Euler-Lagrange's equation

The Euler-Lagrange's equation for our optimization problem has the form

$$\sum_{i=1}^{3}f_i=0 \tag{87}$$

Let $p(x,t)$ be the solution of (70)–(75) for (v^0, y^0). Then, $p(v)$ is defined by transposition, i.e.

$$\left\langle p,y''+Ay\right\rangle=M(y),\qquad\forall y\in D^{-1}_{A+D_t^2}(Q) \tag{88}$$

where

$$M(y)=\left\langle p''+Ap,\,y\right\rangle+\left\langle p,\,l\right\rangle-\left\langle p,\,q\right\rangle-\left\langle p(0),\,y_2\right\rangle+\left\langle p'(0),\,y_1\right\rangle$$

and y satisfies (41)–(46).

We observe that, for given z_d and v, equations (70)–(75) can be solved backward in time starting from $t=T$, i.e. first solving problem (70)–(75) in the subcylinder Q_1, and in turn in Q_{k-1} etc., until the procedure covers the whole cylinder Q. For this purpose, we may apply Theorem 2.

Lemma 2 *Let the hypothesis of Theorem 2 be satisfied. Then, for given* $z_d\in H^{-1,-2}(Q)$, *and any* $v\in L^2(\Sigma)$, *there exists a unique solution*

$$p(v)\in H^{3,3}(Q)\subset\Xi^{3,3}(Q)$$

to the problem (70)–(75) *defined by transposition* (88).

Next we denote by $\overline{y}$ the solution of $P'(\overline{y},\overline{v})=0$ for any fixed $\overline{v}$. Then taking into account (82), (83) and (86) we can express (87) in the form

$$f_2'(\overline{v})=\lambda_0\lambda_1\left\langle\Lambda_1(y^0-z_d),\,\overline{y}\right\rangle_{H^{-1,-2}(Q)}+\lambda_0\lambda_2\left\langle Nv^0,\,\overline{v}\right\rangle_{L^2(\Sigma)}$$
$$\forall(\overline{y},\overline{v})\in RTC(Q_1,(\overline{y},\overline{v})) \tag{89}$$

We transform the first component of the right-hand side of (89) using the formulae (70)–(75). For this purpose setting $v=v_0$ in (70)–(75) and then taking the scalar product of both sides of (70) and (71) by an element $y(v)$ respectively, and then adding both sides of (70) and (71), we get

$$\lambda_0\lambda_1\left\langle\Lambda_1(y^0-z_d),\,\overline{y}\right\rangle_{H^{-1,-2}(Q)}=$$

$$=\left\langle\frac{\partial^2 p}{\partial t^2}+A(t)p,\,\overline{y}\right\rangle_{H^{-1,-2}(Q)}+\left\langle p(x,t+s(t))[1+s'(t)],\,\overline{y}\right\rangle_{H^{-1,-2}[\Omega\times(0,T-h(T))]}$$

$$=\left\langle p,\frac{\partial^2\overline{y}}{\partial t^2}\right\rangle_{H^{-3,-3}(Q)}+\left\langle A(t)p,\,\overline{y}\right\rangle_{H^{-1,-2}(Q)}+$$

$$+\left\langle p(x,t+s(t))[1+s'(t)],\,\overline{y}\right\rangle_{H^{-1,-2}[\Omega\times(0,T-h(T))]}\tag{90}$$

By using the equation (41), the first term on the right-hand side of (90) can be rewritten as

$$\left\langle p,\frac{\partial^2\overline{y}}{\partial t^2}\right\rangle_{H^{-3,-3}(Q)}=-\left\langle p,\,A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)}-\left\langle p,\,\overline{y}(x,t-h(t))\right\rangle_{H^{-1,-2}(Q)}$$

$$=-\left\langle p,\,A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)}-\left\langle p(x,t+s(t))[1+s'(t)],\,\overline{y}\right\rangle_{H^{-1,-2}[\Omega\times(-h(0),T-h(T))]}\tag{91}$$

The second integral on the right-hand side of (90) in view of Green's formula can be expressed as

$$\left\langle A(t)p,\,\overline{y}\right\rangle_{H^{-1,-2}(Q)}=\left\langle p,\,A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)}$$

$$+\left\langle p,\,\frac{\partial\overline{y}}{\partial\eta_A}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}-\left\langle\frac{\partial p}{\partial\eta_A},\,\overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}\tag{92}$$

By using the boundary condition (45), the second term on the right-hand side of (92) can be written as

$$\left\langle p,\frac{\partial\overline{y}}{\partial\eta_A}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}=\left\langle p,\,\overline{y}(x,t-h(t))\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}+\left\langle p,\,G\overline{v}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}$$

$$=\left\langle p(x,t+s(t))[1+s'(t)],\,\overline{y}(x,t)\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(-h(0),T-h(T))]}$$

$$+\left\langle p,\,G\overline{v}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}\tag{93}$$

The last term in (92) can be written as

$$\left\langle\frac{\partial p}{\partial\eta_A},\,\overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}=\left\langle\frac{\partial p}{\partial\eta_A},\,\overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(0,T-h(T))]}$$

$$+\left\langle\frac{\partial p}{\partial\eta_A},\,\overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-h(T),T)]}\tag{94}$$

Substituting (93) and (94) into (92) and then (91) and (92) into (90) we obtain

$$\lambda_0\lambda_1\left\langle\Lambda_1(y^0-z_d),\,\overline{y}\right\rangle_{H^{-1,-2}(Q)}=-\left\langle p,A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)}+\left\langle p,A(t)\overline{y}\right\rangle_{H^{-3,-3}(Q)}$$

$$-\left\langle p(x,t+s(t))[1+s'(t)],\,\overline{y}\right\rangle_{H^{-1,-2}[\Omega\times(-h(0),0)]}$$

$$-\left\langle p(x,t+s(t))[1+s'(t)],\,\overline{y}\right\rangle_{H^{-1,-2}[\Omega\times(0,T-h(T))]}$$

$$+\left\langle p(x,t+s(t))[1+s'(t)],\,\overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(-h(0),0)]}$$

$$+\left\langle p(x,t+s(t))[1+s'(t)],\,\overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(0,T-h(T))]}$$

$$+\left\langle p,\,G\overline{v}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}-\left\langle\frac{\partial p}{\partial\eta_A},\,\overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(0,T-h(T))]}$$

$$-\left\langle\frac{\partial p}{\partial\eta_A},\,\overline{y}\right\rangle_{H^{-5/2}\Xi^{-5/2}[\Gamma\times(T-h(T),T)]}$$

$$+\left\langle p(x,t+s(t))[1+s'(t)],\,\overline{y}\right\rangle_{H^{-1,-2}[\Omega\times(0,T-h(T))]}$$

$$=\left\langle p,\,G\overline{v}\right\rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)}=\left\langle G^{*}p,\,v\right\rangle_{L^2(\Sigma)}\tag{95}$$

Substituting (95) into (89) gives

$$f_2'(\overline{v})=\lambda_0\left\langle G^{*}p+\lambda_2Nv^0,\,\overline{v}\right\rangle_{L^2(\Sigma)}\tag{96}$$

Using the definition of the support functional [4] and dividing both sides of the obtained inequality by λ_0, we finally get

$$\left\langle G^{*}p+\lambda_2Nv^0,\,v-v^0\right\rangle_{L^2(\Sigma)}\geq 0\qquad\forall v\in U_{ad}\tag{97}$$

The last inequality is equivalent to the maximum condition (76).

The uniqueness of the optimal control follows from the strict convexity of the performance functional (62).

This last remark finishes the proof of Theorem 3.

One may also consider analogous optimal control problem with the performance functional

$$\hat{I}(y,v)=\lambda_1\left\|y(v)\mid_\Sigma-z_{\Sigma d}\right\|^2_{H^{-5/2}\Xi^{-5/2}(\Sigma)}+\lambda_2\left\langle(Nv),\,v\right\rangle_{L^2(\Sigma)}\tag{98}$$

where $z_{\Sigma d}$ is a given element in $H^{-5/2}\Xi^{-5/2}(\Sigma)$; we assume that the space $H^{-5/2}\Xi^{-5/2}(\Sigma)$ is such that $y(v)\mid_\Sigma\in H^{-5/2}\Xi^{-5/2}(\Sigma)$. Then the solution of the formulated optimal control problem is equivalent to seeking a pair

$$(y^0,v^0)\in E=D^{-1}_{A+D_t}(Q)\times L^2(\Sigma)$$

that satisfies the equation (41)–(46) and minimizing the cost function (98) with the constraints on controls (63).

We can prove the following theorem:

Theorem 4 *The solution of the optimization problems* (41)–(46), (98), (63) *exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities*:

State equations (41)–(46)

Adjoint equations

$$\frac{\partial^2 p}{\partial t^2}+A(t)p+p\big(x,t+s(t)\big)\big(1+s'(t)\big)=0$$
$$x\in\Omega,\quad t\in\big(0,T-h(T)\big)\tag{99}$$

$$\frac{\partial^2 p}{\partial t^2}+A(t)p=0\qquad x\in\Omega,\quad t\in\big(T-h(T),T\big)\tag{100}$$

$$\frac{\partial p}{\partial\eta_A}=p\big(x,t+s(t)\big)\big(1+s'(t)\big)+\lambda_0\Lambda_2\big(y^0\mid_\Sigma-z_{\Sigma d}\big)$$
$$x\in\Gamma,\quad t\in\big(0,T-h(T)\big)\tag{101}$$

$$\frac{\partial p}{\partial \eta_A} = \lambda_0 \Lambda_2 \left(y^0 \mid_\Sigma - z_{\Sigma d} \right) \quad x \in \Gamma, \quad t \in \left(T - h(T), T \right) \quad (102)$$

$$p(x,T) = 0 \quad x \in \Omega \quad (103)$$

$$\frac{\partial p(x,T)}{\partial t} = 0 \quad x \in \Omega \quad (104)$$

where Λ_2 is a canonical isomorphism of $H^{-5/2}\Xi^{-5/2}(\Sigma)$ into $H^{5/2}\Xi^{5/2}(\Sigma)$.

Maximum condition

$$\left\langle G^* p(v^0) + \lambda_2 N v^0,\, v - v^0 \right\rangle_{L^2(\Sigma)} \geq 0 \quad \forall v \in U_{ad} \quad (105)$$

Moreover, it can be proved the following result.

Lemma 3 *Let the hypothesis of Theorem 2 be satisfied. Then, for given* $z_d \in H^{-5/2}\Xi^{-5/2}(\Sigma)$ *and any* $v \in L^2(\Sigma)$, *there exists a unique solution* $p(v) \in H^{3,3}(Q) \subset \Xi^{3,3}(Q)$ *to the problem* (99)–(104) *defined by transposition* (88).

The idea of the proof of the Theorem 4 is the same as in the case of the Theorem 3.

We must notice that the conditions of optimality derived above (Theorems 3 and 4) allow us to obtain an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on boundary control). It results from the following: the determining of the function $p(x, t)$ in the maximum condition from the adjoint equation is possible if and only if we know that $y^0(x, t)$ will suit the control $v^0(x, t)$. These mutual connections make the practical use of the derived optimization formulas difficult. Thus we resign from the exact determining of the optimal control and we use approximation methods.

In the case of performance functionals (62) and (98) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one [11, 12, 16], which can be solved by the use of the well-known algorithms, e.g. Gilbert's [3, 11, 12, 16] ones.

The practical application of Gilbert's algorithm to optimal control problem for a parabolic system with the boundary condition involving a time lag is presented in [16]. Using of the Gilbert's algorithm a one dimensional numerical example of the plasma control process is solved.

6. Conclusions and perspectives

The derived conditions of the optimality (Theorems 3 and 4) are original from the point of view of application of the Dubovicki–Milutin theorem [4] in solving optimal boundary control problems for second order hyperbolic systems in which time-varying delays appear both in the state equations and in the Neumann boundary conditions.

The existence and uniqueness of solutions for such hyperbolic systems were proved – Lemma 1 and Theorem 2. The optimal control was characterized by using the adjoint equations – Lemmas 2 and 3. Necessary and sufficient conditions of optimality with the quadratic performance functionals (62) and (98) and constrained control (63) are derived for the Neumann problem – Theorems 3 and 4.

The proved optimization results (Theorems 3 and 4) constitute a novelty of the paper with respect to the references [10, 12] concerning application of the Lions scheme [19] for solving linear quadratic hyperbolic problems of optimal control.

The results presented in the paper can be treated as a generalization of the results obtained in [14] to the case of additional time-varying lags appearing in the state equations.

Moreover, the optimization problems presented here constitute a generalization of optimal control problems considered in [13] to the case of hyperbolic systems with time-varying lags appearing both in the state equations and in the Neumann boundary conditions.

The obtained optimization theorems (Theorems 3 and 4) demand the assumption dealing with the non-empty interior of the set Q_2 representing the inequality constraints.

Therefore, we approximate the set Q_2 by the regular admissible cone (if $\operatorname{int} Q_2 = \varnothing$, then this cone does not exist).

It is worth mentioning that the obtained results can be reinforced by omitting the assumption concerning the non-empty interior of the set Q_2 and utilizing the fact that the equality constraints in the form of the hyperbolic equations are "decoupling".

The optimal control problem reduces to seeking $v^0 \in Q_2'$ and minimizing the performance index $I(v)$. Then we approximate the set Q_2' representing the inequality constraints by the regular tangent cone and for the performance index $I(v)$ we construct the regular improvement cone.

Making use of the Dubovicki–Milutin method the similar conditions of the optimality can be derived for hyperbolic systems with the Neumann boundary conditions involving incommensurate time lags.

The proposed methodology based on the Dubovicki–Milutin scheme can be presented as a specific case study concerning hyperbolic problems described by partial differential equations of the hyperbolic type involving time lags appeared in the integral form both in the state equations and in the Neumann boundary conditions.

The same procedure can be applied to solving optimal control problems for nonlinear hyperbolic systems.

Another direction of research will be numerical examples concerning the determination of optimal control with constraints for time-varying delay hyperbolic systems.

Appendix

For the functions $y \in H^{r,s}(Q)$, we shall formulate three central theorems. For this purpose we shall define the following spaces

i. The space $W(a, b)$ ([20], Vol. 1, p. 10) Let a and b be two real numbers, finite or not, $a < b$. Let X, Y be Hilbert spaces and m be an integer ≥ 1.

We set

$$W(a,b) = \left\{ y \mid y \in L^2(a,b;X), \quad \frac{\mathrm{d}^m y}{\mathrm{d}t^m} = y^{(m)} \in L^2(a,b;X) \right\} \quad (A1)$$

which is a Hilbert space provided with the form

$$\|y\|_{W(a,b)} = \left(\|y\|^2_{L^2(a,b;X)} + \left\|y^{(m)}\right\|^2_{L^2(a,b;Y)} \right)^{\frac{1}{2}} \quad (A2)$$

ii. The space $B(a,\ b;\ E)$ ([20], Vol. 1, p. 19).

Generally, if E is a Hilbert space, we set

$$B(a,b;E)=\begin{cases} C^0\left([a,b;E]\right) = \text{continuous functions of } [a,b]\to E \\ \text{if } a \text{ and } b \text{ are finite;} \\ \\ \text{continuous bounded functions of } t\geq a\to E \\ \text{if } a \text{ is finite and } b=+\infty; \\ \\ \text{continuous bounded functions of } R\to E \\ \text{if } a=-\infty,\ b=+\infty \end{cases} \quad \text{(A3)}$$

We provide $B(a,\ b;\ E)$ with the norm

$$\|y\|_{B(a,b;E)} = \sup_{t\in(a,b)} \|y(t)\|_E \quad \text{(A4)}$$

Consequently, using the results of ([20], Vol. 1, p. 19) we can formulate theorem.

Theorem 3.1. ([20], Vol. 1, p. 19): With the notation (A3) for $y \in W(a,b)$ we have

$$y^{(j)} \in B\left(a,b;[X,Y]_{(j+\frac{1}{2})/m}\right), \quad 0\leq j\leq m-1 \quad \text{(A5)}$$

Then $y \to y^{(j)}$ being a continuous and linear mapping of

$$W(a,b)\to B\left(a,b;[X,Y]_{(j+\frac{1}{2})/m}\right).$$

In ([20], Vol. 1, p. 43) the following interpolation theorem is proved.

Theorem 9.6. ([20], Vol. 1, p.43): Assume that the boundary Γ of Ω is a $(n-1)$ dimensional infinitely differentiable variety, Ω being locally on one side of Γ. Moreover, in general we assume that Ω is bounded. Then

$$\left[H^{s_1}(\Omega), H^{s_2}(\Omega)\right]_\Theta = H^{(1-\Theta)s_1+\Theta s_2}(\Omega) \quad \text{(A6)}$$

for all $s_1 > 0$, $s_2 < s_1$, $0 < \Theta < 1$ (with equivalent norms).

For purposes of application one of the most useful results is the Trace Theorem. Consequently, using the results of ([20], Vol. 2, p. 9) we may formulate the following theorem.

Theorem 2.1. ([20], Vol. 2, p. 9): For $y \in H^{r,s}(Q)$ with $r > \frac{1}{2}$, $s \geq 0$, we may define:

$$\begin{cases} \dfrac{\partial^j y}{\partial \eta_A^j} \text{ on } \Sigma \qquad \text{if } j < r-\dfrac{1}{2} \text{ (integer } j\geq 0) \\ \\ \dfrac{\partial^j y}{\partial \eta_A^j} \in H^{\mu_j,v_j}(\Sigma) \end{cases} \quad \text{(A7)}$$

where

$$\frac{\mu_j}{r} = \frac{v_j}{s} = \frac{r-j-\frac{1}{2}}{r} \quad (v_j = 0 \text{ if } s=0) \quad \text{(A8)}$$

Then $y \to \dfrac{\partial^j y}{\partial \eta_A^j}$ are continuous linear mappings of $H^{r,s}(Q) \to H^{\mu_j,v_j}(\Sigma)$.

Afterwards, we shall present the following theorems.

Let K will denote the cone of directions of decrease for the functional $F(x)$ at the point x_0.

Theorem 7.5 [4]: If $F(x)$ is differentiable at x_0, then $F(x)$ is regularly decreasing at x_0 and $K = \{h : (F'(x_0),\ h) < 0\}$.

Theorem 9.1 [4] (Lusternik): Let $P(x)$ be an operator mapping E_1 into E_2, differentiable in a neighbourhood of a point x_0, $P(x_0) = 0$. Let $P'(x)$ be a continuous in a neighbourhood of x_0, and suppose that $P'(x_0)$ maps E_1 onto E_2 (i.e., the linear equation $P'(x_0)h = b$ has a solution h for any $b \in E_2$). Then the set of tangent directions K to the set $Q = \{x : P(x) = 0\}$ at the point x_0 is the subspace $K = \{h : P'(x_0)h = 0\}$.

Theorem 10.1 [4]: Let K be a subspace.
Then $K^* = \{f \in E' : f(x) = 0 \text{ for all } x \in K\}$ (this set is sometimes known as the annihilator of K).

Theorem 10.2 [4]: Let $f \in E'$, $K_1 = \{x : f(x) = 0\}$, $f_2 = \{x : f(x) \geq 0\}$, $K_3 = \{x : f(x) > 0\}$.
Then $K_1^* = \{\lambda f, -\infty < \lambda < \infty\}$, $K_2^* = \{\lambda f, 0 \leq \lambda < \infty\}$, $K_3^* = E'$ for $f = 0$ and $K_3^* = K_2^*$ for $f \neq 0$.

Let Q be a set in a topological linear space E, $x_0 \in Q$, K_b the cone of feasible directions for Q at x_0 and K_k the cone of tangent directions for Q at x_0. In turns out that in some cases the duals of these cones coincide with the set of linear functionals which are supports for the Q at the point x_0. Denote the latter set by Q^*, i.e., $Q^* = \{f \in E' : f(x) \geq f(x_0) \text{ for all } x \in Q\}$.

Theorem 10.5 [4]: Let Q be a closed convex set. Then $K_k^* = Q^*$. If moreover $Q^0 \neq \Phi$, then $K_b^* = Q^*$.

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References

1. Bahaa G.M., Kotarski W., *Optimality conditions for n × n infinite-order parabolic coupled systems with control constraints and general performance index*, "IMA Journal of Mathematical Control and Information", Vol. 25, No. 1, 2008, 49–57, DOI: 10.1093/imamci/dnm003.
2. Dubovitski A.Ya., Milyutin A.A., *Extremum problems with certain constraints*, "Doklady Akademii Nauk SSSR", Vol. 149, No. 4, 1963, 759–762.
3. Gilbert E.S., *An iterative procedure for computing the minimum of a quadratic form on a convex set*, "SIAM Journal of Control", Vol. 4, No. 1, 1966, 61–80, DOI: 0.1137/0304007.
4. Girsanov I.V., *Lectures on the Mathematical Theory of Extremal Problems*, Publishing House of the University of Moscow, 1970.
5. Kotarski W., *Further generalization of the Dubovicki-Milutin theorem*, "Journal of Optimization Theory and Applications", Vol. 54, No. 3, 1987, 565–573. DOI: 10.1007/BF00940202.
6. Kotarski W., *Optimal control of a system governed by a parabolic equation with an infinite number of variables*, "Journal of Optimization Theory and Applications", Vol. 60, No. 1, 1989, 33–41, DOI: 10.1007/BF00938797.

7. Kotarski W., *Optimal control of a system governed by a parabolic equation with an infinite number of variables and time delay*, "Journal of Optimization Theory and Applications", Vol. 63, No. 1, 1989, 57–67, DOI: 10.1007/BF00940731.
8. Kotarski W., Bahaa G.M., *Optimal control problem for infinite order hyperbolic system with mixed control-state constraints*, "European Journal of Control", Vol. 11, No. 2, 2005, 150–156, DOI: 10.3166/ejc.11.150-156.
9. Kotarski W., El-Saify H.A., Bahaa G.M., *Optimal control of parabolic equation with an infinite number of variables for non-standard functional and time delay*, "IMA Journal of Mathematical Control and Information", Vol. 19, No. 4, 2002, 461–476, DOI: 10.1093/imamci/19.4.461.
10. Kowalewski A., *Boundary control problem of a hyperbolic systems with time lags*, "IMA Journal of Mathematical Control and Information", Vol. 10, No. 2, 1993, 261–272, DOI: 10.1093/imamci/10.3.261.
11. Kowalewski A., *Optimal control of a distributed hyperbolic system with time-varying lags*, "International Journal of Control", Vol. 71, No. 3, 1998, 419–435, DOI: 10.1080/002071798221759.
12. Kowalewski A., *Optimal Control of Infinite Dimensional Distributed Parameter Systems with Delays*, AGH University of Krakow, Krakow, 2001.
13. Kowalewski A., *Extremal problems for time lag hyperbolic systems*, Proceedings of the 25th International Conference on Methods and Models in Automation and Robotics (MMAR). Międzyzdroje, Poland, 2021, 245–250, DOI: 10.1109/MMAR49549.2021.9528456.
14. Kowalewski A., *Extremal problems for hyperbolic systems with boundary conditions involving time-varying delays*, Proceedings of the 26th International Conference on Methods and Models in Automation and Robotics (MMAR). Międzyzdroje, Poland, 2022, 122–127, DOI: 10.1109/MMAR55195.2022.9874285.
15. Kowalewski A., Extremal problems for second order hyperbolic systems with multiple time delays, "Archives of Control Sciences", Vol. 33, No. 1, 2023, 101–126, DOI: 10.24425/acs.2023.
16. Kowalewski A., Duda J., *On some optimal control problem for a parabolic system with boundary condition involving a time-varying lag*, "IMA Journal of Mathematical Control and Information", Vol. 9, No. 2, 1992, 131–146, DOI: 10.1093/imamci/9.2.131.
17. Lasiecka I., *Conical approximations of sets with applications to optimization problems*, PhD thesis. Technical University, Warsaw, 1974.
18. Lasiecka I., *Generalization of the Dubovitzky-Milutyn optimality conditions*, "Journal of Optimization Theory and Applications", Vol. 24, No. 3, 1978, 421–436, DOI: 10.1007/BF00932886.
19. Lions J.L., *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, 1971.
20. Lions J.L., Magenes E., *Non-Homogeneous Boundary Value Problems and Applications*, Vols. 1 and 2, Springer-Verlag, Berlin, 1972.
21. Łędzewicz-Kowalewska U., *On some specification of the Dubovitskii-Milyutin method*, "Nonlinear Analysis: Theory, Methods & Applications", Vol. 10, No. 12, 1986, 1367–1371, DOI: 10.1016/0362-546X(86)90107-0.
22. Łędzewicz-Kowalewska U., *Application of the method of contractor directions to the Dubovitskii-Milyutin formalism*, "Journal of Mathematical Analysis and Applications", Vol. 125, No. 1, 1987, 174–184, DOI: 10.1016/0022-247X(87)90172-7.
23. Łędzewicz-Kowalewska U., *Optimal control problems of hyperbolic systems with operator and nonoperator equality constraints*, "Applicable Analysis", Vol. 27, No. 1-3, 1988, 199–215, DOI: 10.1016/0362-546X(88)90028-4.
24. Łędzewicz-Kowalewska U., *On the optimal control problems with Gâteaux differentiable operator constraints*, "Journal of Mathematical Analysis and Applications", Vol. 130, No. 2, 1988, 301–315, DOI: 10.1016/0022- 247X(88)90307-1.
25. Łędzewicz-Kowalewska U., *Application of some specification of the Dubovitskii-Milyutin method to problems of optimal control*, "Nonlinear Analysis: Theory, Methods & Applications", Vol. 12, No. 2, 1988, 101–108, DOI: 10.1080/00036818808839733.
26. Łędzewicz-Kowalewska U., *Euler-Lagrange equation in the case of nonregular equality constraints*, "Journal of Optimization Theory and Applications", Vol. 71, No. 3, 1991, 549–568, DOI: 10.1007/BF00941403.
27. Łędzewicz-Kowalewska U., *On the optimal control problems of parabolic equations with an infinite number of variables and with equality constraints*, "Differential and Integral Equations", Vol. 4, No. 2, 1991, 369–381, DOI: 10.57262/die/1371569618.
28. Łędzewicz U., *Extension of the local maximum principle to abnormal optimal control problems*, "Journal of Optimization Theory and Applications", Vol. 77, No. 3, 1993, 661–681, DOI: 10.1007/BF00940455.
29. Łędzewicz U., *On distributed parameter control systems in the abnormal case and in the case of nonoperator equality constraints*, "Journal of Applied Mathematics and Stochastic Analysis", Vol. 6, No. 2, 1993, 137–151, DOI: 10.1155/S1048953393000139.
30. Łędzewicz-Kowalewska U., *The extremum principle for some types of distributed parameter control systems*, "Applicable Analysis", Vol. 48, No. 1-4, 1993, 1–21, DOI: 10.1080/00036819308840146.
31. Maslov V.P., *Operators Methods*, Russian. "Science", Moscov, 1973.
32. Neustadt L.W., *An abstract variational theory with applications to a broad class of optimization problems. I. General theory*, "SIAM Journal on Control", Vol. 4, No. 3, 1966, 505–527, DOI: 10.1137/0304040.
33. Neustadt L.W., *An abstract variational theory with applications to a broad class of optimization problems. II. Applications*, "SIAM Journal on Control", Vol. 5, No. 1, 1967, 90–137, DOI: 10.1137/0305007.
34. El-Saify H.A., Bahaa G.M., *Optimal control for $n{\times}n$ hyperbolic systems involving operators of infinite order*, "Mathematica Slovaca", Vol. 52, No. 4, 2002, 409–424.
35. Walczak S., *The local extremum principle for problems with constraints upon phase coordinates*, Vol. XXXII. 3. Société des Sciences et des Lettres de Łódź, 1982.
36. Walczak S., *On some control problem*, "Acta Universitatis Lodziensis. Folia Mathematica", 1984, 187–196.
37. Walczak S., *Some properties of cones in normed spaces and their application to investigating extremal problems*, "Journal of Optimization Theory and Applications", Vol. 42, No. 4, 1984, 561–582, DOI: 10.1007/BF00934567.

Problemy ekstremalne dla systemów hiperbolicznych drugiego rzędu ze zmiennymi opóźnieniami czasowymi

Streszczenie: Zaprezentowano ekstremalne problemy dla systemów hiperbolicznych ze zmiennymi opóźnieniami czasowymi. Rozwiązano problem optymalnego sterowania brzegowego dla systemów hiperbolicznych drugiego rzędu, w których zmienne opóźnienia czasowe występują zarówno w równaniach stanu oraz w warunkach brzegowych typu Neumanna. Tego rodzaju równania stanowią w liniowym przybliżeniu uniwersalny model matematyczny dla procesów fizycznych, w których ma miejsce przesyłanie sygnałów na odległość, takich jak linie długie typu elektrycznego, hydraulicznego i innych. Korzystając z metody Dubowickiego-Milutina wyprowadzono warunki konieczne i wystarczające optymalności dla problemu liniowo-kwadratowego.

Słowa kluczowe: sterowanie brzegowe, systemy hiperboliczne drugiego rzędu, zmienne opóźnienia czasowe

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