Extremal Problems for Infinite Order Parabolic Systems with Multiple Time-Varying Lags

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Abstract: Extremal problems for infinite order parabolic systems with multiple time-varying lags are presented. An optimal boundary control problem for infinite order parabolic systems in which multiple time-varying lags appear in the state equations and in the boundary condition simultaneously is solved. The time horizon is fixed. Making use of Dubovicki–Milutin scheme, necessary and sufficient conditions of optimality for the Neumann problem with the quadratic performance functionals and constrained control are derived.

Keywords: boundary control, infinite order, parabolic systems, multiple time-varying lags

1. Introduction

Extremal problems are now playing an ever-increasing role in applications of mathematical control theory. It has been discovered that notwithstanding the great diversity of these problems, then can be approached by a unified functional-analytic approach, first suggested by Dubovicki and Milutin. The general theory of extremal problems has been developed so intensely recently that its basic concepts may now be considered complete.

Extremal problems were the object of mathematical research at the very earliest stages of the development of mathematics.

The first results were then systematized and brought together under the heading of the calculus of variations with its innumerable applications to physics, automatic control, and mechanics.

Technological progress presented the calculus of variations with a new type of problem – the control of objects whose control parameters are varied in some closed set with boundary. Quite varied problems of this type were investigated by Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko, who established a necessary condition for an extremum the so-called Pontryagin maximum principle.

The nature of this condition and the form of the optimal solutions were so different from the classical theorems of the calculus of variations that popular science writers began to speak of the advent of a new calculus of variations.

In 1962 Dubovicki and Milutin found a necessary condition for an extremum in the form of an equation set down in the language of functional analysis. They were able to derive, as special cases of this condition, almost all previously known necessary extremum conditions and thus to recover the lost theoretical unity of the calculus of variations.

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The Dubovicki–Milutin method has been applied to solve many sophisticated mathematical problems. For this purpose, we shall present the following papers.

The main object of the paper [13] is to extend the Dubovicki–Milutin results and to show that the necessary optimality conditions can be derived for a broader class of optimisation problems [12, 13]. To do that, a type of conical approximation weaker than those employed by Neustadt [27, 28] and Dubovicki–Milutin [4] is introduced; and it leads to a stronger kind of separation theorem [12, 13]. Using this theorem, the necessary optimality conditions given by Neustadt [27, 28] and Dubovicki–Milutin [4] are generalised [12, 13]. The theoretical considerations are illustrated by an example.

In the paper [16], the Dubovicki–Milutin method presented in [6] is applied to obtain a necessary condition for a problem with only one equality constraint. In [31] some generalisation of the Dubovicki–Milutin method is shown for the case of n equality constraints in any form under the same assumptions about the tangent cones and the cones dual to them. This generalisation is applied in [29] and [30]. Consequently, in the paper [16] a specification of the Dubovicki–Milutin method is obtained without those assumptions from [31], but in the case of n equality constraints given in the operator form.

In the paper [17], the method of contractor direction obtained by Altman is to applied to the Dubovicki–Milutin formalism. By using this method, some specification of the Dubovicki–Milutin theorem is proved for the case of n equality constraints given by Gâteaux – differentiable operators.

Next in the paper [18] two kinds of optimal control problems of hyperbolic systems with additional equality constraints are considered: a problem with the operator equality constraint in the form of terminal condition and a problem with nonoperator equality constraint in the form $u(\cdot) \in U$, where U – some set. The extremum principles are proved: for the first problem – by using some specification of the Dubovicki–Milutin method in the case of n equality constraints in the operator form and for the second one – by using some generalisation of the Dubovicki–Milutin theory.

In the paper [19] the problem of optimal control with mixed equality and inequality operator constraints is considered under the assumption of Gâteaux differentiability. The extremum principle for this kind of problem is obtained by using some specification of the Dubovicki–Milutin method [16] for the case of n equality constrains given by Gâteaux differentiable operators.

A some specification of the Dubovicki–Milutin method is given by without any additional assumption about the cones but for the case on n equality constraints given in the operator form in [16]. This specification is applied in the paper [20] to obtain a necessary condition for the problem of an optimal control with equality constrains on the phase coordinates and the control.

In the paper [21], an optimisation problem with inequality and equality constraints in Banach spaces is considered in the case when the operators which determine the equality constraints are nonregular. In this case, the classical Euler–Lagrange equation has degenerate form, i.e., does not depend on the functional to be minimized. Applying some generalisation of the Lusternik theorem to the Dubovicki–Milutin method, the family of Euler–Lagrange equations is obtained in the nondegenerate form under the assumption of twice Fréchet differentiability of the operators. The Pareto–optimal problem is also considered.

In the paper [22], optimal control problems of systems governed by parabolic equations with an infinite number of variables and with additional equality constraints are considered. The extremum principles as well as sufficient conditions of optimality are formulated by using certain extensions of the Dubovicki–Milutin method.

In the paper [23], an optimal control problem with terminal data is considered in the so–called abnormal case, i.e., when the classical Pontryagin–type maximum principle has a degenerate form which does not depend on the functional being minimized. An extension of the Dubovicki–Milutin method to the nonregular case, obtained by applying Avakov's generalisation of the Lusternik theorem, is presented. By using this extension, a local maximum principle which has a nondegenerate form also in the abnormal case is proved. An example which supports the theory is given.

In the paper [24], a general distributed parameter control problem in Banach spaces with integral cost functional and with given initial and terminal data is considered. An extension of the Dubovicki–Milutin method to the case of nonregular operator equality constraints, based on Avakov's generalisation of the Lusternik theorem, is presented. This result is applied to obtain an extension of the Extremum Principle for the case of abnormal optimal control problems. Then a version of this problem with nonoperator equality constraints is discussed and the Extremum Principle for this problem is presented.

In the paper [25], two types of general distributed parameter control problems in the Banach space with integral cost functional, with given terminal data and with initial data, given or insufficient, are considered. By using of the Dubovicki–Milutin method and some of its extensions, the extremum principles as well as sufficient conditions of optimality are proved for both types of problems.

Distributed parameter systems with delays can be used to describe many phenomena in the real world. As is well known, heat conduction, properties of elastic-plastic material, fluid dynamics, diffusion-reaction processes, transmission of the signals at the certain distance by using electric long lines, etc., all lie within this area. The object that we are studying (temperature, displacement, concentration, velocity, etc.) is usually referred to as the state.

We are interested in the case where the state satisfies proper differential equations that are derived from basic physical laws, such as Newton's law, Fourier's law, etc. The space in which the state exists is called the state space, and the equation that the state satisfies is called the state equation. In particular, we are interested in the cases where the state equations are one of the following types: partial differential equations, integro-differential equations, or abstract evolution equations.

Extremal problems for multiple time-varying lag infinite order parabolic systems are investigated. The purpose of this paper is to show the use of Dubovicki–Milutin method in solving optimal control problems for infinite order parabolic systems in which multiple time-varying lags appear both in the state equations and in the Neumann boundary conditions.

As an example, an optimal boundary control problem for a system described by a linear infinite order partial differential equation of parabolic type with different multiple time-varying lags appearing both in the state equation and in the Neumann boundary condition is considered. Such an equation constitutes in a linear approximation universal mathematical model for many diffusion processes.

The performance functional has the quadratic form. The time horizon is fixed. Finally, we impose same constraints on the boundary control. Making use of the Dubovicki–Milutin theorem, necessary and sufficient conditions of optimality with the quadratic performance functionals and constrained control are derived for the Neumann problem.

2. Principal Notations

R	– set of real numbers.
R^n	– real n-dimensional euclidean space.
$\Omega \subset R^n$	– bounded, open set with smooth boundary Γ.
$x = \{x_1, \ldots, x_n\}$	– denotes the space variable such that $x \subset \Omega \subset R^n$.
t	– denotes time variable such that $t \in (0, T),\ T < \infty$.
$Q = \Omega \times (0, T)$	– cylinder.
$\Sigma = \Gamma \times (0, T)$	– boundary of the cylinder Q.
v	– boundary control.
v^0	– optimal boundary control.
$y(x,t;\ v)$	– denotes the state variable corresponding to a given control v.
$p(x,t;\ v)$	– denotes the adjoint variable corresponding to a given control v.
$I(v)$	– denotes the cost function.
$X,\ Y$	– are the Hilbert spaces.
$X',\ Y'$	– denote the dual spaces of X and Y respectively.
$A,\ B$	– denote operators.
$A^*,\ B^*$	– denote the adjoint operators to A and B respectively.
$\mathscr{D}(A)$	– domain of the operator A.
$\lVert \cdot \rVert_X$	– norm in the space X.
$< \cdot, \cdot >_X$	– denotes scalar product defined in the space X.
$\mathscr{L}(X,\ Y)$	– space of linear operators on X into Y.

3. Main function spaces

$H^m(a,b;X)$ – is a Sobolev space of order m ($m \geq 0$ and is integer) of functions defined on $(a,\ b)$ and taking value in X and such that

$$H^m(a,b;X) = \left\{ y \mid y,\, y^{(1)} = \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots, y^{(m)} = \frac{\mathrm{d}^m y}{\mathrm{d}t^m} \in L^2(a,b;X) \right\}$$

with the scalar product

$$\left\langle y, \varphi \right\rangle_{H^m(a,b;X)} = \sum_{j=0}^{n} \int_a^b \left\langle y^{(j)}(t), \varphi^{(j)}(t) \right\rangle_X \mathrm{d}t$$

$H^m(a,b;X)$ – is a Hilbert space.

$L^2(a, b; X)$ – is a space of measurable functions y defined on (a, b) and taking values in X and such that

$$\|y\|_{L^2(a,b;X)} = \left(\int_a^b \|y(t)\|_X^2 \, \mathrm{d}t\right)^{\frac{1}{2}} < \infty$$

where: X is a Hilbert space.

Finally, for any pair or real numbers $r,\ s \geq 0$, we introduce the Sobolev space $H^{r,s}(Q)$ – a Sobolev space ([15], Vol. 2, p. 6) defined by

$$H^{r,s}(Q) = H^0\left(0,T;H^r(\Omega)\right) \cap H^s\left(0,T;H^0(\Omega)\right)$$

which is a Hilbert space normed by

$$\left(\int_0^T \|y(t)\|^2_{H^r(\Omega)} \, \mathrm{d}t + \|y\|^2_{H^s(0,T;H^0(\Omega))}\right)^{\frac{1}{2}}$$

where: $H^r(\Omega)$ and $H^s(0,T;H^0(\Omega))$ are defined as a interpolation of the following spaces

$$\left[H^m(\Omega), H^0(\Omega)\right]_\Theta = H^r(\Omega)$$

where: $(1 - \Theta)m = r,\ 0 < \Theta < 1,$

$$\left[H^m\left(0,T;H^0(\Omega)\right), H^0\left(0,T;H^0(\Omega)\right)\right]_\Theta = H^s\left(0,T;H^0(\Omega)\right)$$

where: $(1 - \Theta)m = s,\ 0 < \Theta < 1.$

4. The Dubovicki–Milutin theorem [6]

The Dubovicki–Milutin theorem arises from the geometric form of the Hahn-Banach theorem (a theorem about the separation of convex sets).

Let us assume that

E – a linear topological space, locally convex

$I(x)$ – a functional defined on E

$A_i,\ i = 1, 2, \ldots, n$ – sets in E with inner points which represent inequality constraints

B – a set in E without inner points representing equality constraint.

We must find some conditions necessary for a local minimum of the functional $I(x)$ on the set $Q = \bigcap_{i=1}^n A_i \cap B$, or find a point $x_0 \in E$, so that $I(x_0) = \min\limits_{Q \cap U} I(x)$, where U means a certain environment of the point x_0.

We define the set $A_0 = \{x : I(x) < I(x_0)\}$.

Then we formulate the necessary condition of optimality as follows: in the environment of the local minimum point, the intersection of system of sets (the set on which the functional attains smaller values than $I(x_0)$ and the sets representing constraints) is empty or $\bigcap_{i=0}^n A_i \cap B = \varnothing$.

The condition $\bigcap_{i=0}^n A_i \cap B = \varnothing$ is also equivalent to the one in which there are approximations of the sets $A_i,\ i = 1, 2, \ldots, n$ and B instead of A_i or B ones. These approximations are cones with the vertex in a point $\{0\}$.

We shall approximate the inequality constraints by the regular admissible cones $RAC(A_i, x_0)$ $(i = 1, 2, \ldots, n)$, the equality constraint by the regular tangent cone $RTC(B, x_0)$ and for the performance functional we shall construct the regular improvement cone $RFC(I, x_0)$.

Then we have the necessary condition of the optimality $I(x)$ on the set $Q = \bigcap_{i=1}^n A_i \cap B$ in the form of Euler-Lagrange's equation $\sum_{i=1}^{n+1} f_i = 0$, where f_i $(i = 1, 2, \ldots, n + 1)$ are linear, continuous functionals, all of them are not equal to zero at the same time and they belong to the adjoint cones

$$f_i \in \left[RAC(A_i, x_0)\right]^*, \quad i = 1, 2, \ldots, n$$

$$f_{i+1} \in \left[RTC(B, x_0)\right]^*, \quad f_0 \in \left[RFC(I, x_0)\right]^*$$

$$\left\{f_i \in \left[RAC(A_i, x_0)\right]^* \Leftrightarrow f_i(x) \geq 0 \quad \forall x \in RAC(A_i, x_0)\right\}$$

For convex problems, i.e. problems in which the constraints are convex sets and the performance functional is convex, the Euler-Lagrange equation is the necessary and sufficient condition of optimality if only certain additional assumptions are fulfilled (the so-called Slater's condition).

5. Preliminaries

Let Ω be a bounded open set of R^n with smooth boundary Γ. We define the infinite order Sobolev space $H^\infty\{a_\alpha, 2\}(\Omega)$ of functions $\Phi(x)$ defined on Ω [1, 2] as follows

$$H^\infty\{a_\alpha, 2\} = \left\{\Phi(x) \in C^\infty(\Omega) : \sum_{|\alpha|=0}^\infty a_\alpha \left\|\mathscr{D}^\alpha \Phi\right\|_2^2 < \infty\right\} \tag{1}$$

where $C^\infty(\Omega)$ is a space of infinite differentiable functions, $a_\alpha \geq 0$ is a numerical sequence and $\|\cdot\|_2$ is a norm in the space $L^2(\Omega)$, and

$$\mathscr{D}^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \ldots (\partial x_n)^{\alpha_n}}, \tag{2}$$

where: $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index for differentiation, $|\alpha| = \sum_{i=1}^n \alpha_i$.

The space $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ [1, 2] is defined as the formal conjugate space to the space $H^\infty\{a_\alpha, 2\}(\Omega)$, namely:

$$H^{-\infty}\{a_\alpha, 2\}(\Omega) = \left\{\Psi(x) : \Psi(x) = \sum_{|\alpha|=0}^\infty (-1)^{|\alpha|} a_\alpha \mathscr{D}^\alpha \Psi_\alpha(x)\right\} \tag{3}$$

where $\Psi_\alpha \in L^2(\Omega)$ and $\sum_{|\alpha|=0}^\infty a_\alpha \left\|\Psi_\alpha\right\|_2^2 < \infty$.

The duality pairing of the spaces $H^\infty\{a_\alpha, 2\}(\Omega)$ and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ is postulated by the formula

$$\left\langle \Phi, \Psi \right\rangle = \sum_{|\alpha|=0}^\infty a_\alpha \int_\Omega \Psi_\alpha(x) \mathscr{D}^\alpha \Phi(x) dx, \tag{4}$$

where: $\Phi \in H^\infty\{a_\alpha, 2\}(\Omega)$, $\Psi \in H^{-\infty}\{a_\alpha, 2\}(\Omega)$.

From above, $H^\infty\{a_\alpha, 2\}(\Omega)$ is everywhere dense in $L^2(\Omega)$ with topological inclusions and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ denotes the topological dual space with respect to $L^2(\Omega)$ so we have the following chain:

$$H^{\infty}\{a_\alpha,2\}(\Omega) \subseteq L^2(\Omega) \subseteq H^{-\infty}\{a_\alpha,2\}(\Omega). \tag{5}$$

6. Existence and uniqueness of solutions

Consider now the distributed parameter system described by the following parabolic equation

$$\frac{\partial y}{\partial t} + A(t)y + \sum_{i=1}^{m} y\left(x, t - h_i(t)\right) = u \quad x \in \Omega, \quad t \in (0,T) \tag{6}$$

$$y(x,t') = \Phi_0(x,t') \quad x \in \Omega, \quad t' \in \left[-\Delta(0), 0\right) \tag{7}$$

$$\frac{\partial y}{\partial \eta_A} = \sum_{s=1}^{l} y\left(x, t - k_s(t)\right) + v \quad x \in \Gamma, \quad t \in (0,T) \tag{8}$$

$$y(x,t') = \Psi_0(x,t') \quad x \in \Gamma, \quad t' \in \left[-\Delta(0), 0\right) \tag{9}$$

$$y(x,0) = y_0(x) \quad x \in \Omega \tag{10}$$

where: Ω has the same properties as in the Section 5.

$$y \equiv y(x,t;v), \quad u \equiv u(x,t), \quad v \equiv v(x,t),$$

$$Q = \Omega \times (0,T), \quad \bar{Q} = \Omega \times [0,T], \quad Q_0 = \Omega \times \left[-\Delta(0), 0\right)$$

$$\Sigma = \Gamma \times (0,T), \quad \Sigma_0 = \Gamma \times \left[-\Delta(0), 0\right)$$

$h_i(t)$, $k_s(t)$ are functions representing multiple time-varying lags, Φ_0 is an initial function defined on Q_0, Ψ_0 is an initial function defined on Σ_0.

Moreover,

$$\Delta(0) = \max\left\{h_1(0), h_2(0), \ldots h_m(0), k_1(0), k_2(0), \ldots k_l(0)\right\}$$

The operator $\dfrac{\partial}{\partial t} + A$ in the state equation (6) is an infinite order parabolic operator and A is given by

$$Ay = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathscr{D}^{2\alpha} y(x,t) \tag{11}$$

and

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathscr{D}^{2\alpha} \tag{12}$$

is an infinite order elliptic partial differential operator [3].

The equations (6)–(10) constitute a Neumann problem [3]. The left-hand side of (8) is written in the following form

$$\frac{\partial y}{\partial \eta_A} = \sum_{|w|=0}^{\infty} \left(\mathscr{D}^{w} y(v)\right) \cos(n, x_i) = q(x,t)$$
$$x \in \Gamma, \quad t \in (0,T) \tag{13}$$

where $\dfrac{\partial}{\partial \eta_A}$ is a normal derivative at Γ, directed towards the exterior of Ω, $\cos(n, x_i)$ is an i-th direction cosine of n, n-being the normal at Γ exterior to Ω and

$$q(x,t) = \sum_{s=1}^{l} y\left(x, t - k_s(t)\right) + v(x,t) \tag{14}$$

Let $t - h_i(t)$ for $i = 1, \ldots, m$ and $t - k_s(t)$ for $s = 1, \ldots, l$ be strictly increasing functions, $h_i(t)$ and $k_s(t)$ being non-negative in $[0, T]$ and also being a C^1 functions. Then, there exist the inverse functions of $t - h_i(t)$ and $t - k_s(t)$ respectively.

Let us denote $r_i(t) \triangleq t - h_i(t)$ and $\lambda_s(t) \triangleq t - k_s(t)$, then the inverse functions of $r_i(t)$ and $\lambda_s(t)$ have the form $t = f_i(r_i) = r_i + s_i(r_i)$ and $t = \varepsilon_s(r_s) = r_s + q_s(r_s)$, where $s_i(r_i)$ and $q_s(r_s)$ are time-varying predictions.

Let $f_i(t)$ and $\varepsilon_s(t)$ be the inverse functions of $t - h_i(t)$ and $t - k_s(t)$ respectively. Thus we define the following iteration:

$$\hat{t}_0 = 0$$

$$\hat{t}_1 = \min\left\{f_1(0), f_2(0), \ldots, f_m(0), \varepsilon_1(0), \varepsilon_2(0), \ldots \varepsilon_l(0)\right\}$$

$$\hat{t}_2 = \min\left\{f_1(\hat{t}_1), f_2(\hat{t}_1), \ldots, f_m(\hat{t}_1), \varepsilon_1(\hat{t}_1), \varepsilon_2(\hat{t}_1), \ldots \varepsilon_l(\hat{t}_1)\right\}$$

$$\vdots$$

$$\hat{t}_j = \min\left\{f_1(\hat{t}_{j-1}), f_2(\hat{t}_{j-1}), \ldots, f_m(\hat{t}_{j-1}), \varepsilon_1(\hat{t}_{j-1}), \varepsilon_2(\hat{t}_{j-1}), \ldots \varepsilon_l(\hat{t}_{j-1})\right\}$$

First we shall prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (6)–(10) for the case where the boundary control $v \in L^2(\Sigma)$. For this purpose we introduce the Sobolev space $H^{\infty,1}(Q)$ ([15], Vol. 2, p. 6) defined by

$$H^{\infty,1}(Q) = H^0\left(0,T; H^{\infty}\{a_\alpha,2\}(\Omega)\right) \cap H^1\left(0,T; H^0(\Omega)\right) \tag{15}$$

which is a Hilbert space normed by

$$\left(\int_0^T \left\|y(t)\right\|^2_{H^{\infty}\{a_\alpha,2\}(\Omega)} dt + \left\|y\right\|^2_{H^1(0,T;H^0(\Omega))}\right)^{\frac{1}{2}} \tag{16}$$

where: the space $H^1(0, T; H^0(\Omega))$ is defined in Chapter 1 of [15], Vol. 1 respectively.

The existence of a unique solution for the mixed initial-boundary value problem (6)–(10) on the cylinder Q can be proved using a constructive method, i.e., first, solving (6)–(10) on the subcylinder Q_1 and in turn on Q_2, etc. until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

For simplicity, we introduce the notation

$$E_j \triangleq \left(\hat{t}_{j-1}, \hat{t}_j\right), \quad Q_j = \Omega \times \Sigma_j, \quad Q_0 = \Omega \times \left[-\Delta(0), 0\right)$$

$$\Sigma = \Gamma \times E_j, \quad \Sigma_0 = \Gamma \times \left[-\Delta(0), 0\right) \quad \text{for } j = 1, \ldots$$

Using the Theorem 15.2 of [15] (Vol. 2, p. 81) we can prove the following lemma.

Lemma 1 *Let*

$$u \in \left(H^{\infty,1}(Q)\right)', \quad v \in L^2(\Sigma) \tag{17}$$

$$f_j \in \left(H^{\infty,1}(Q)\right)' \tag{18}$$

where

$$f_j(x,t) = u(x,t) - \sum_{i=1}^{m} y_{j-1}\left(x, t - h_i(t)\right)$$

$$y_{j-1}\left(\cdot,(j-1)a\right) \in H^{\infty}\{a_\alpha, 2\}(\Omega) \tag{19}$$

$$q_j \in L^2(\Sigma_j) \tag{20}$$

where

$$q_j(x,t) = \sum_{s=1}^{l} y_{j-1}\left(x, t - k_s(t)\right) + v(x,t)$$

Then, there exists a unique solution $y_j \in H^{\infty,1}(Q_j)$ for the mixed initial-boundary value problem (6), (10), (19).

Proof: We observe that for $j = 1$,

$$\sum_{i=1}^{m} y_{j-1}|_{Q_0}\left(x, t - h_i(t)\right) = \sum_{i=1}^{m} \Phi_0\left(x, t - h_i(t)\right)$$

and

$$\sum_{s=1}^{l} y_{j-1}|_{\Sigma_0}\left(x, t - k_s(t)\right) = \sum_{s=1}^{l} \Psi_0\left(x, t - k_s(t)\right)$$

respectively. Then the assumptions (18), (19) and (20) are fulfilled if we assume that $\Phi_0 \in H^{\infty,1}(Q_0)$, and $\Psi_0 \in L^2(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in H^{\infty,1}(Q_1)$. In order to extend the result to Q_2, we have to prove that $y_1\left(\cdot,\hat{t}_1\right) \in H^{\infty}\{a_\alpha,2\}(\Omega)$, $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$ and $f_2 \in \left(H^{\infty,1}(Q_2)\right)'$. Really, from the Theorems 2.1 and 2.2 [7] $y_1 \in H^{\infty,1}(Q_1)$ implies that the mapping $t \to y_1(\cdot,t)$ is continuous from $[0,\hat{t}_1] \to H^{\infty}\{a_\alpha,2\}(\Omega)$. Thus, $y_1(\cdot,\hat{t}_1) \in H^{\infty}\{a_\alpha,2\}(\Omega)$. Then using the Trace Theorem (Theorem 2.3 of [7]) we can verify that $y_1 \in H^{\infty,1}(Q_1)$ implies that $y_1 \to y_1|_{\Sigma_1}$ is a linear, continuous mapping of $H^{\infty,1}(Q_1) \to H^{\infty,1}(\Sigma_1)$. Thus, $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$. Also it is easy to notice that the assumption (18) follows from the fact that $y_1 \in H^{\infty,1}(Q_1)$ and $u \in \left(H^{\infty,1}(Q)\right)'$. Then, there exists a unique solution $y_2 \in H^{\infty,1}(Q_2)$. The foregoing result is now summarized for $j = 3, \ldots$.

Theorem 1 Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 \in H^{\infty}\{a_\alpha,2\}(\Omega)$, $\Phi_0 \in H^{\infty,1}(Q_0)$, $\Psi_0 \in L^2(\Sigma_0)$, $v \in L^2(\Sigma)$ and $u \in \left(H^{\infty,1}(Q)\right)'$. Then, there exists a unique solution $y \in H^{\infty,1}(Q)$ for the mixed initial-boundary value problem (6)–(10). Moreover, $y(\cdot,\hat{t}_j) \in H^{\infty}\{a_\alpha,2\}(\Omega)$ for $j = 3, \ldots$.

We refer to Lions and Magenes ([15], Vol. 2) for the definition and properties of $H^{r,s}$ and $(H^{r,s})'$ respectively.

In the sequal, we shall fix $u \in \left(H^{\infty,1}(Q)\right)'$.

7. Problem formulation. Optimization theorems

We shall restrict our considerations to the case of the boundary control. Therefore we shall formulate the optimal control problem in the context of Theorem 1.

Let us denote by $Y = H^{\infty,1}(Q)$ the space of states and by $U = L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem.

The performance functional is given by

$$I(v) = \lambda_1 \int_Q \left| y(x,t;v) - z_d \right|^2 dxdt + \lambda_2 \int_0^T \int_\Gamma (Nv)v \, d\Gamma \, dt \tag{21}$$

where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 > 0$; z_d is a given element in $L^2(Q)$ and N is a strictly positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$. We note from Theorem 1 that for any $v \in U_{ad}$ the cost function (21) is well-defined since $y(v) \in H^{\infty,1}(Q) \subset L^2(Q)$.

We assume the following constraints on controls:

$$v \in U_{ad} \text{ is a closed, convex subset of } U \text{ with non-empty interior, a subset of } U \tag{22}$$

The optimal control problem (6)–(10), (21), (22) will be solved as the optimization one in which the function v is the unknown function.

Making use of the Dubovicki–Milutin theorem [10] we shall derive the necessary and sufficient conditions of optimality for the optimization problem (6)–(10), (21), (22).

The solution of the stated optimal control problem is equivalent to seeking a pair $(y^0, v^0) = H^{\infty,1}(Q) \times L^2(\Sigma)$ that satisfies the equation (6)–(10) and minimizing performance functional (21) with the constraints on control (22).

We formulate the necessary and sufficient conditions of the optimality in the form of Theorem 2.

Theorem 2 *The solution of the optimization problem* (6)–(10), (21), (22) *exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:*

$$\frac{\partial y^0}{\partial t} + A(t)y^0 + \sum_{i=1}^{m} y^0\left(x, t - h_i(t)\right) = u \quad (x,t) \in \Omega \times (0,T) \tag{23}$$

$$y^0(x,t') = \Phi_0(x,t') \quad (x,t') \in \Omega \times \left[-\Delta(0), 0\right) \tag{24}$$

$$y^0(x,0) = y_0(x) \quad x \in \Omega \tag{25}$$

$$\frac{\partial y^0}{\partial \eta_A} = \sum_{s=1}^{l} y^0\left(x, t - k_s(t)\right) + v^0 \quad (x,t) \in \Gamma \times (0,T) \tag{26}$$

$$y^0(x,t') = \Psi_0(x,t') \quad (x,t') \in \Gamma \times \left[-\Delta(0), 0\right) \tag{27}$$

Adjoint equations

$$-\frac{\partial p}{\partial t} + A^*(t)p + \sum_{i=1}^{m} p\left(x, t + s_i(t)\right)\left(1 + s'(t)\right) = \\ = \lambda_1(y^0 - z_d) \; (x,t) \in \Omega \times \left(0, T - \Delta(T)\right) \tag{28}$$

$$-\frac{\partial p}{\partial t} + A^*(t)p = \lambda_1(y^0 - z_d) \quad (x,t) \in \Omega \times \left(T - \Delta(T), T\right) \tag{29}$$

$$\frac{\partial p}{\partial \eta_{A^*}} = \sum_{s=1}^{l} p\left(x, t + q_s(t)\right)\left(1 + q_s'(t)\right) \; (x,t) \in \Gamma \times \left(0, T - \Delta(t)\right) \tag{30}$$

$$\frac{\partial p}{\partial \eta_{A^*}} = 0 \quad (x,t) \in \Gamma \times \left(T - \Delta(t), T\right) \tag{31}$$

$$p(x,T) = 0 \quad x \in \Omega \tag{32}$$

Maximum condition

$$\int_0^T \int_\Gamma (p + \lambda_2 N v^0)(v - v^0)\, d\Gamma\, dt \geq 0 \quad \forall v \in U_{ad} \tag{33}$$

where:

$$\begin{cases} \Delta(T) = \max\left\{h_1(T), h_2(T), \ldots, h_m(T), k_1(T), k_2(T), \ldots, k_l(T)\right\} \\ \dfrac{\partial p(v)}{\partial \eta_{A^*}}(x,t) = \sum_{|w|=0}^{\infty} \left(\mathscr{D}^w p(v)\right)\cos(n, x_i) \\ A^* p = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \mathscr{D}^{2\alpha} p(x,t) \end{cases} \tag{34}$$

Outline of the proof:
According to the Dubovicki–Milutin theorem [10], we approximate the set representing the inequality constraints by the regular admissible cone, the equality constraint by the regular tangent cone and the performance functional by the regular improvement cone.

a) Analysis of the equality constraint

The set Q_1 representing the equality constraint has the form

$$Q_1 = \left\{ \begin{array}{ll} \dfrac{\partial y}{\partial t} + A(t)y + \sum_{i=1}^{m} y\left(x, t - h_i(t)\right) = u & (x,t) \in \Omega \times (0,T) \\ y(x,t') = \Phi_0(x,t') & (x,t') \in \Omega \times \left[-\Delta(0), 0\right) \\ y(x,0) = y_0(x) & x \in \Omega \\ \dfrac{\partial y}{\partial \eta_A} = \sum_{s=1}^{l} y\left(x, t - k_s(t)\right) + v & (x,t) \in \Gamma \times (0,T) \\ y(x,t') = \Psi_0(x,t') & (x,t') \in \Gamma \times \left[-\Delta(0), 0\right) \end{array} \right\} \tag{35}$$

We construct the regular tangent cone of the set Q_1 using the Lusternik theorem (Theorem 9.1 [6]). For this purpose, we define the operator P in the form

$$\begin{aligned} P(y,v) = \Bigg(& \frac{\partial y}{\partial t} + Ay + \sum_{i=1}^{m} y\left(x, t - h_i(t)\right) - u, \\ & y(x,t') - \Phi_0(x,t'), \quad y(x,0) - y_0(x), \\ & \frac{\partial y}{\partial \eta_A} - \sum_{s=1}^{l} y\left(x, t - k_s(t)\right) - v, \quad y(x,t') - \Psi_0(x,t') \Bigg) \end{aligned} \tag{36}$$

The operator P is the mapping from the space $H^{\infty,1}(Q) \times L^2(\Sigma)$ into the space

$$\left(H^{\infty,1}(Q)\right)' \times H^{\infty,1}(Q_0) \times H^{\infty}\{a_\alpha, 2\}(\Omega) \times L^2(\Sigma) \times L^2(\Sigma_0).$$

The Fréchet differential of the operator P can be written in the following form:

$$\begin{aligned} P'(y^0, v^0)(\overline{y}, \overline{v}) = \Bigg(& \frac{\partial \overline{y}}{\partial t} + A\overline{y} + \sum_{i=1}^{m} \overline{y}\left(x, t - h_i(t)\right), \quad \overline{y}\,|_{Q_0}\ (x,t'), \\ & \frac{\partial \overline{y}}{\partial \eta_A} - \sum_{s=1}^{l} \overline{y}\left(x, t - k_s(t)\right) - \overline{v}, \quad \overline{y}\,|_{\Sigma_0}\ (x,t') \Bigg) \end{aligned} \tag{37}$$

Really, $\dfrac{\partial}{\partial t}$ (Theorem 2.8 [26]), A(t) (Theorem 2.1 [14]) and $\dfrac{\partial}{\partial \eta_A}$ (Theorem 2.3 [15]) are linear and bounded mappings.

Using Theorem 1, we can prove that P' is the operator "one to one" from the space $H^{\infty,1}(Q) \times L^2(\Sigma)$ onto the space

$$\left(H^{\infty,1}(Q)\right)' \times H^{\infty,1}(Q_0) \times H^{\infty}\{a_\alpha, 2\}(\Omega) \times L^2(\Sigma) \times L^2(\Sigma_0).$$

Considering that the assumptions of the Lusternik's theorem are fulfilled, we can write down the regular tangent cone for the set Q_1 in a point $(y^0,\ v^0)$ in the form

$$RTC\left(Q_1, (y^0, v^0)\right) = \left\{(\overline{y}, \overline{v}) \in E, \quad P'(y^0, v^0)(\overline{y}, \overline{v}) = 0\right\} \tag{38}$$

It is easy to notice that it is a subspace. Therefore, using Theorem 10.1 [6] we know the form of the functional belonging to the adjoint cone

$$f_1(\overline{y}, \overline{v}) = 0 \quad \forall (\overline{y}, \overline{v}) \in RTC\left(Q_1, (y^0, v^0)\right) \tag{39}$$

b) Analysis of the constraint on controls

The set $Q_2 = Y \times U_{ad}$ representing the inequality constraints is a closed and convex one with non-empty interior in the space E.

Using Theorem 10.5 [6] we find the functional belonging to the adjoint regular admissible cone, i.e.

$$f_2(\overline{y}, \overline{v}) \in \left[RTC\left(Q_2, (y^0, v^0)\right)\right]^*$$

We can note if E_1, E_2 are two linear topological spaces, then the adjoint space to $E = E_1 \times E_2$ has the form

$$E^* = \left\{f = (f_1, f_2); \quad f_1 \in E_1^*, \quad f_2 \in E_2^*\right\}$$

and

$$f(x) = f_1(x_1) + f_2(x_2)$$

So we note the functional $f_2(\overline{y}, \overline{v})$ as follows

$$f_2(\overline{y}, \overline{v}) = f_1'(\overline{y}) + f_2'(\overline{v}) \tag{40}$$

where: $f_1'(\overline{y}) = 0 \quad \forall y \in Y$ (Theorem 10.1 [6]), $f_2'(\overline{v})$ is a support functional to the set U_{ad} in a point v_0 (Theorem 10.5 [6]).

c) Analysis of the performance functional

Using Theorem 7.5 [6] we find the regular improvement cone of the performance functional (21)

$$RTC\left(I, (y^0, v^0)\right) = \left\{(\overline{y}, \overline{v}) \in E, \quad I'(y^0, v^0)(\overline{y}, \overline{v}) < 0\right\} \tag{41}$$

where $I'(y^0, v^0)(\overline{y}, \overline{v})$ is the Fréchet differential of the performance functional (21) and it can be written as

$$I'(y^0,v^0)(\overline{y},\overline{v}) = 2\lambda_0\lambda_1\int_Q (y^0 - z_d)\overline{y}\,dxdt + 2\lambda_0\lambda_2\int_0^T\int_\Gamma (Nv^0)\overline{v}\,d\Gamma dt.$$

On the basis of Theorem 10.2 [6] we find the functional belonging to the adjoint regular improvement cone, which has the form

$$\lambda_3(\overline{y},\overline{v}) = -\lambda_0\lambda_1\int_Q (y^0 - z_d)\overline{y}\,dxdt - \lambda_0\lambda_2\int_0^T\int_\Gamma (Nv^0)\overline{v}\,d\Gamma dt \tag{42}$$

where $\lambda_0 > 0$.

d) Analysis of Euler-Lagrange's equation

The Euler-Lagrange's equation for our optimization problem has the form

$$\sum_{i=1}^{3} f_i = 0 \tag{43}$$

Let $p(x,\,t)$ be the solution of (28)–(32) for $(v^0,\,y^0)$ and denote by y the solution of $P'(\overline{y},\overline{v}) = 0$ for any fixed $\overline{v}$. Then taking into account (39), (40) and (42) we can express (43) in the form

$$f_2'(\overline{v}) = \lambda_0\lambda_1\int_Q (y^0 - z_d)\overline{y}\,dxdt + \lambda_0\lambda_2\int_0^T\int_\Gamma (Nv^0)\overline{v}\,d\Gamma dt$$
$$\forall(\overline{y},\overline{v}) \in RTC\left(Q_1,(\overline{y},\overline{v})\right). \tag{44}$$

We transform the first component of the right-hand side of (44) introducing the adjoint variable by adjoint equations (28)–(32).

For this purpose, multiplying both sides of (28)–(29) by $\overline{y}$, then integrating over $\Omega\times\left(0,T-h(T)\right)$ and $\Omega\times\left(T-h(T),T\right)$ respectively, and then adding both sides of (28)–(29), we get

$$\begin{aligned}
\lambda_0\lambda_1\int_Q (y^0 - z_d)\overline{y}\,dxdt &= \lambda_0\int_Q \left(-\frac{\partial p}{\partial t} + A^*p\right)\overline{y}\,dxdt + \\
&+ \lambda_0\sum_{i=1}^{m}\int_0^T\int_\Omega p\left(x,t+s_i(t)\right)\left(1+s_i'(t)\right)\overline{y}\,dxdt = \\
&= \lambda_0\int_Q p\frac{\partial y}{\partial t}dxdt + \lambda_0\int_Q A^*p\overline{y}\,dxdt + \\
&+ \lambda_0\sum_{i=1}^{m}\int_0^{T-\Delta(T)}\int_\Omega p\left(x,t+s_i(t)\right)\left(1+s_i'(t)\right)\overline{y}\,dxdt
\end{aligned} \tag{45}$$

Using the equation (6), the first integral on the right-hand side of (45) can be written as

$$\begin{aligned}
\lambda_0\int_Q p\frac{\partial\overline{y}}{\partial t}dxdt &= \\
&= -\lambda_0\int_Q pA\overline{y}\,dxdt - \lambda_0\sum_{i=1}^{m}\int_0^T\int_\Omega p(x,t)\overline{y}\left(x,t-h_i(t)\right)dxdt = \\
&= -\lambda_0\int_Q pA\overline{y}\,dxdt - \\
&- \lambda_0\sum_{i=1}^{m}\int_{-h_i(0)}^{T-h_i(T)}\int_\Omega p\left(x,t_i+s_i(t_i)\right)\left(1+s_i'(t_i)\right)\overline{y}(x,t_i)\,dxdt_i
\end{aligned} \tag{46}$$

where $t_i = t - h_i(t),\ t = t_i + s_i(t_i)$ and $dt = [1+s_i'(t_i)]dt_i$.

The second integral on the right-hand side of (45) in view of Green's formula can be expressed as

$$\lambda_0\int_Q A^*p\overline{y}\,dxdt = \lambda_0\int_Q pA\overline{y}\,dxdt + \lambda_0\int_0^T\int_\Gamma p\frac{\partial\overline{y}}{\partial\eta_A}d\Gamma dt - \lambda_0\int_0^T\int_\Gamma \frac{\partial p}{\partial\eta_{A^*}}\overline{y}\,d\Gamma dt \tag{47}$$

Using the boundary condition (8), the second term on the right-hand side of (47) can be written as

$$\begin{aligned}
\lambda_0\int_0^T\int_\Gamma p\frac{\partial\overline{y}}{\partial\eta_A}d\Gamma dt &= \lambda_0\sum_{s=1}^{l}\int_0^T\int_\Gamma p(x,t)\left[\overline{y}\left(x,t-k_s(t)\right)+\overline{v}\right]d\Gamma dt = \\
&= \lambda_0\sum_{s=1}^{l}\int_{-k_s(0)}^{T-k_s(T)}\int_\Gamma p\left(x,t_s+q_s(t)\right)\left(1+q_s'(t_s)\right)\overline{y}(x,t_s)\,d\Gamma dt_s + \\
&+ \lambda_0\int_0^T\int_\Gamma p\overline{v}\,d\Gamma dt = \\
&= \lambda_0\sum_{s=1}^{l}\int_{-k_s(0)}^{0}\int_\Gamma p\left(x,t_s+q_s(t_s)\right)\left(1+q_s'(t_s)\right)\overline{y}(x,t_s)\,d\Gamma dt_s + \\
&+ \lambda_0\sum_{s=1}^{l}\int_0^{T-k_s(T)}\int_\Gamma p\left(x,t_s+q_s(t_s)\right)\left(1+q_s'(t_s)\right)\overline{y}(x,t_s)\,d\Gamma dt_s + \\
&+ \lambda_0\int_0^T\int_\Gamma p\overline{v}\,d\Gamma dt
\end{aligned} \tag{48}$$

where $t_s = t - k_s(t),\ t = t_s + q_s(t_s)$ and $dt = [1+q_s'(t_s)]dt_s$.

The last term in (47) can be rewritten as

$$\lambda_0\int_0^T\int_\Gamma \frac{\partial p}{\partial\eta_{A^*}}\overline{y}\,d\Gamma dt = \lambda_0\int_0^{T-\Delta(T)}\int_\Gamma \frac{\partial p}{\partial\eta_{A^*}}\overline{y}\,d\Gamma dt + \lambda_0\int_{T-\Delta(T)}^{T}\int_\Gamma \frac{\partial p}{\partial\eta_{A^*}}\overline{y}\,d\Gamma dt \tag{49}$$

Substituting (48), (49) into (47) and then (46), (47) into (45) we obtain

$$\begin{aligned}
&\lambda_0\lambda_1\int_Q (y^0 - z_d)\overline{y}\,dxdt = \\
&= -\lambda_0\int_Q pA\overline{y}\,dxdt - \lambda_0\sum_{i=1}^{m}\int_{-h_i(0)}^{0}\int_\Omega p\left(x,t+s_i(t)\right)\left(1+s_i'(t)\right)\overline{y}\,dxdt - \\
&- \lambda_0\sum_{i=1}^{m}\int_0^{T-h_i(T)}\int_\Omega p\left(x,t+s_i(t)\right)\left(1+s_i'(t)\right)\overline{y}\,dxdt + \lambda_0\int_Q pA\overline{y}\,dxdt + \\
&+ \lambda_0\sum_{s=1}^{l}\int_{-k_s(0)}^{0}\int_\Gamma p\left(x,t+q_s(t)\right)\left(1+q_s'(t)\right)\overline{y}\,d\Gamma dt + \\
&+ \lambda_0\sum_{s=1}^{l}\int_0^{T-k_s(T)}\int_\Gamma p\left(x,t+q_s(t)\right)\left(1+q_s'(t)\right)\overline{y}\,d\Gamma dt + \lambda_0\int_0^T\int_\Gamma p\overline{v}\,d\Gamma dt - \\
&- \lambda_0\int_0^{T-\Delta(T)}\int_\Gamma \frac{\partial p}{\partial\eta_{A^*}}\overline{y}\,d\Gamma dt - \lambda_0\int_{T-\Delta(T)}^{T}\int_\Gamma \frac{\partial p}{\partial\eta_{A^*}}\overline{y}\,d\Gamma dt + \\
&+ \lambda_0\sum_{i=1}^{m}\int_0^{T-\Delta(T)}\int_\Omega p\left(x,t+s_i(t)\right)\left(1+s_i'(t)\right)\overline{y}\,dxdt = \\
&= \lambda_0\int_0^T\int_\Gamma p\overline{v}\,d\Gamma dt
\end{aligned} \tag{50}$$

Substituting (50) into (44) gives

$$f_2'(\overline{v}) = \lambda_0\int_0^T\int_\Gamma (p + \lambda_2 Nv^0)\overline{v}\,d\Gamma dt. \tag{51}$$

Using the definition of the support functional [6] and dividing both members of the obtained inequality by λ_0, we finally get

$$\int_0^T\int_\Gamma (p+\lambda_2 Nv^0)(v-v^0)\,d\Gamma dt \geq 0 \quad \forall v \in U_{ad} \tag{52}$$

The last inequality is equivalent to the maximum condition (33).

In order to prove the sufficiency of the derived conditions of the optimality, we use the fact that constraints and the performance functional are convex and that the Slater's condition is satisfied (Theorem 15.3 [6]). Then, there exists a point $(\tilde{y},\tilde{v}) \in \operatorname{int} Q_2$ such that $(\tilde{y},\tilde{v}) \in Q_1$.

This fact follows immediately from the existence of non-empty interior of the set Q_2 and from the existence of the solution of the equation (6)–(10) as well.

This last remark finishes the proof of Theorem 2.

One may also consider analogous optimal control problem with the performance functional

$$\hat{I}(y,v) = \lambda_1 \int_\Sigma \left| y(v)\,|_\Sigma - z_{\Sigma d}\right|^2 d\Gamma dt + \lambda_2 \int_0^T\int_\Gamma (Nv)v\,d\Gamma dt \tag{53}$$

where: $z_{\Sigma d}$ is a given element in $L^2(\Sigma)$.

From Theorem 1 and the trace theorem ([15], vol. 2, p. 9) for such $v \in L^2(\Sigma)$, there exists a unique solution $H^{\infty,1}(Q)$ with $y\,|_\Sigma \in L^2(\Sigma)$. Thus $\hat{I}(y,v)$ is well-defined. Then the solution of the formulated optimal control problem is equivalent to seeking a pair $(y^0,v^0) \in E = H^{\infty,1}(Q) \times L^2(\Sigma)$ that satisfies the equation (6)–(10) and minimizing the cost function (53) with the constraints on controls (22).

We can prove the following theorem:

Theorem 3 *The solution of the optimization problems (6)–(10), (53), (22) exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:*
State equations (23)–(27).

Adjoint equations

$$\begin{aligned} -\frac{\partial p}{\partial t} + A^*(t)p + \sum_{i=1}^{m} p\left(x,t+s_i(t)\right)\left(1+s_i'(t)\right) &= \\ = 0 \quad (x,t) \in \Omega \times \left(0, T-\Delta(T)\right)& \end{aligned} \tag{54}$$

$$-\frac{\partial p}{\partial t} + A^*(t)p = 0 \quad (x,t) \in \Omega \times \left(T-\Delta(T), T\right) \tag{55}$$

$$\begin{aligned} \frac{\partial p}{\partial \eta_{A^*}} &= \\ = \sum_{s=1}^{l} p\left(x,t+q_s(t)\right)&\left(1+q_s'(t)\right) + \lambda_1(y^0 - z_{\Sigma_d}) \quad (x,t) \in \Gamma \times \left(0, T-\Delta(t)\right) \end{aligned} \tag{56}$$

$$\frac{\partial p}{\partial \eta_{A^*}} = \lambda_1(y^0 - z_{\Sigma_d}) \quad (x,t) \in \Gamma \times \left(T-\Delta(t), T\right) \tag{57}$$

$$p(x,T) = 0 \quad x \in \Omega \tag{58}$$

Maximum condition

$$\int_0^T\int_\Gamma (p+\lambda_2 Nv^0)(v-v^0)\,d\Gamma dt \quad \forall v \in U_{ad} \tag{59}$$

The idea of the proof of the Theorem 3 is the same as in the case of the Theorem 2.

Remark 1 *The coupled system (23)–(27) with (54)–(58) corresponds to the case of the observation on the boundary for the optimal control problem (6)–(10) with (22) and (53).*

Remark 2 *The existence of a unique solution for the adjoint problem (54)–(58) on the cylinder Q can be proved using a constructive method. It is easy to notice that for given z_{Σ_d} and v, the problem (54)–(58) can be solved backwards in time starting from $t = T$, i.e. first, solving (54)–(58) on subcylinder Q_K and in turn on Q_{K-1}, etc. until the procedure covers the whole cylinder Q. For this purpose, we may apply Theorem 1 (with an obvious change of variables) to the adjoint problem (54)– (58) (with reversed sense of time, i.e. $t' = T - t$). Then, for given $z_{\Sigma_d} \in L^2(\Sigma)$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v) \in H^{\infty,1}(Q)$ for the adjoint problem (54)–(58).*

We must notice that the conditions of optimality derived above (Theorems 2 and 3) allow us to obtain an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on boundary control). It results from the following: the determining of the function $p(x,t)$ in the maximum condition from the adjoint equation is possible if and only if we know that $y^0(x,t)$ will suit the control $v^0(x,t)$. These mutual connections make the practical use of the derived optimization formulas difficult. Thus we resign from the exact determining of the optimal control and we use approximation methods.

In the case of performance functionals (21) and (53) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one [7, 8] which can be solved by the use of the well-known algorithms, e.g. Gilbert's [5, 7, 8] ones.

The practical application of Gilbert's algorithm to optimal control problem for a parabolic system with boundary conditions involving multiple time-varying lags is presented in [8]. Using of the Gilbert's algorithm a one dimensional numerical example of the plasma control process is solved ([8], 98–107).

8. Conclusions and perspectives

The derived conditions of the optimality (Theorems 2 and 3) are original from the point of view of application of the Dubovicki–Milutin theorem for solving optimal control problems for infinite order parabolic systems in which different multiple time-varying lags appear both in the state equations and in the Neumann boundary conditions. The existence and uniqueness of solutions for such infinite order parabolic systems were proved – Lemma 1 and Theorem 1.

The results presented in the paper can be treated as a generalization of the results obtained in [9] to the case of infinite order parabolic systems with multiple time-varying lags.

The proved optimization results (Theorems 2 and 3) constitute a novelty of the paper with respect to references [7] and [8] concerning application of the Lions scheme [14] for solving linear quadratic parabolic problems for the case of the Neumann problem.

Moreover, the optimization problems presented here constitute a generalization of optimal control problems considered in [10] for parabolic systems with constant time lags appearing in the state equations and in the boundary conditions simultaneously.

The obtained optimization theorems (Theorems 2 and 3) demand the assumption dealing with the non-empty interior of the set Q_2 representing the inequality constraints.

Therefore, we approximate the set Q_2 by the regular admissible cone (if $\operatorname{int} Q_2 = \varnothing$, then this cone does not exist).

It is worth mentioning that the obtained results can be reinforced by omitting the assumption concerning the non-empty interior of the set Q_2 and utilizing the fact that the equality constraints in the form of the parabolic equations are "decoupling". The optimal control problem reduces to seeking $v_0 \in Q_2'$ and minimizing the performance index $I(v)$. Then we approximate the set Q_2' representing the inequality constraints by the regular tangent cone and for the performance index $I(v)$ we construct the regular improvement cone.

One may also derive the necessary and sufficient conditions of optimality for infinite order parabolic system with the Neumann boundary conditions involving integral time lags [11].

Finally, one may consider more complex optimization problems with non-differentiable and non-continuous performance functionals.

According to the author similar optimal control problems can be solved for infinite order hyperbolic systems.

Another direction of future research will be sophisticated numerical examples concerning the determination of optimal control with constraints for infinite order parabolic systems with multiple time-varying lags.

The ideas mentioned above will be developed in forthcoming papers.

Appendix

For the functions $y \in H^{r,s}(\Omega)$, we shall formulate three central theorems. For this purpose we shall define the following spaces

i. The space $W(a,\ b)$ ([15], Vol. 1, p. 10) Let a and b be two real numbers, finite or not, $a < b$. Let X, Y be Hilbert spaces and m be an integer ≥ 1.

We set

$$W(a,b) = \left\{ y \mid y \in L^2(a,b;X), \quad \frac{\mathrm{d}^m y}{\mathrm{d}t^m} = y^{(m)} \in L^2(a,b;X) \right\} \tag{A1}$$

which is a Hilbert space provided with the form

$$\|y\|_{W(a,b)} = \left(\|y(t)\|^2_{L^2(a,b;X)} + \|y^{(m)}\|^2_{L^2(a,b;Y)} \right)^{\frac{1}{2}} \tag{A2}$$

ii. The space $B(a,\ b;\ E)$ ([15], Vol. 1, p. 19).

Generally, if E is a Hilbert space, we set

$$B(a,b;E) = $$
$$= \begin{cases} C^0\left([a,b;E]\right) = \text{continuous functions of } [a,b] \to E \text{ if } a \text{ and } b \text{ are finite;} \\ \text{continuous bounded functions of } t \geq a \to E \text{ if } a \text{ is finite and } b = +\infty; \\ \text{continuous bounded functions of } R \to E \text{ if } a = -\infty, b = +\infty \end{cases} \tag{A3}$$

We provide $B(a,\ b;\ E)$ with the norm

$$\|y\|_{B(a,b;E)} = \sup_{t \in (a,b)} \|y(t)\|_E \tag{A4}$$

Consequently, using the results of ([15], Vol. 1, p. 19) we can formulate theorem.

Theorem 2.1 [7]**:** With the notation (A3) for $y \in W(a,b)$ we have

$$y^{(j)} \in B\left(a,b;[X,Y]_{(j+\frac{1}{2})/m}\right), \quad 0 \leq j \leq m-1 \tag{A5}$$

Then $y \to y^{(j)}$ being a continuous and linear mapping of

$$W(a,b) \to B\left(a,b;[X,Y]_{(j+\frac{1}{2})/m}\right).$$

In ([15], Vol. 1, p. 43) the following interpolation theorem is proved.

Theorem 2.2 [7]**:** Assume that the boundary Γ of Ω is a $(n-1)$ dimensional infinitely differentiable variety, Ω being locally on one side of Γ. Moreover, in general we assume that Ω is bounded. Then

$$\left[H^{s_1}(\Omega), H^{s_2}(\Omega)\right]_\Theta = H^{(1-\Theta)s_1+\Theta s_2}(\Omega) \tag{A6}$$

for all $s_1 > 0$, $s_2 < s_1$, $0 < \Theta < 1$ (with equivalent norms).

For purposes of application one of the most useful results is the Trace Theorem. Consequently, using the results of ([15], Vol. 2, p. 9) we may formulate the following theorem.

Theorem 2.3 [7]**:** For $y \in H^{r,s}(Q)$ with $r > \dfrac{1}{2}$, $s \geq 0$, we may define:

$$\begin{cases} \dfrac{\partial^j y}{\partial \eta_A^j} \text{ on } \Sigma \qquad \text{if } j < r - \dfrac{1}{2} \text{ (integer } j \geq 0) \\ \dfrac{\partial^j y}{\partial \eta_A^j} \in H^{\mu_j, v_j}(\Sigma) \end{cases} \tag{A7}$$

where

$$\frac{\mu_j}{r} = \frac{v_j}{s} = \frac{r - j - \frac{1}{2}}{r} \quad (v_j = 0 \text{ if } s = 0) \tag{A8}$$

Then $y \to \dfrac{\partial^j y}{\partial \eta_A^j}$ are continuous linear mappings of

$H^{r,s}(Q) \to H^{\mu_j, v_j}(\Sigma)$.

Afterwards, we shall present the following theorems.

Let K will denote the cone of directions of decrease for the functional $F(x)$ at the point x_0.

Theorem 7.5 [6]**:** If $F(x)$ is differentiable at x_0, then $F(x)$ is regularly decreasing at x_0 and $K = \{h : (F'(x_0),\ h) < 0\}$.

Theorem 9.1 [6] **(Lusternik):** Let $P(x)$ be an operator mapping E_1 into E_2, differentiable in a neighbourhood of a point x_0, $P(x_0) = 0$. Let $P'(x)$ be a continuous in a neighbourhood of x_0, and suppose that $P'(x_0)$ maps E_1 onto E_2 (i.e., the linear equation $P'(x_0)h = b$ has a solution h for any $b \in E_2$). Then the set of tangent directions K to the set $Q = \{x : P(x) = 0\}$ at the point x_0 is the subspace $K = \{h : P'(x_0)h = 0\}$.

Theorem 10.1 [6]: Let K be a subspace. Then $K^* = \{f \in E' : f(x) = 0 \text{ for all } x \in K\}$ (this set is sometimes known as the annihilator of K).

Theorem 10.2 [6]: Let $f \in E'$, $K_1 = \{x : f(x) = 0\}$, $f_2 = \{x : f(x) \geq 0\}$, $K_3 = \{x : f(x) > 0\}$.
Then $K_1^* = \{\lambda f, -\infty < \lambda < \infty\}$, $K_2^* = \{\lambda f, 0 \leq \lambda < \infty\}$, $K_3^* = E'$ for $f = 0$ and $K_3^* = K_2^*$ for $f \neq 0$.

Let Q be a set in a topological linear space E, $x_0 \in Q$, K_b the cone of feasible directions for Q at x_0 and K_k the cone of tangent directions for Q at x_0. In turns out that in some cases the duals of these cones coincide with the set of linears functionals which are supports for the Q at the point x_0. Denote the latter set by Q^*, i.e., $Q^* = \{f \in E' : f(x) \geq f(x_0) \text{ for all } x \in Q\}$.

Theorem 10.5 [6]: Let Q be a closed convex set. Then $K_k^* = Q^*$. If moreover $Q^0 \neq \Phi$, then $K_b^* = Q^*$.

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Problemy ekstremalne dla systemów parabolicznych nieskończonego rzędu z wielokrotnymi zmiennymi opóźnieniami czasowymi

Streszczenie: Zaprezentowano ekstremalne problemy dla systemów parabolicznych nieskończonego rzędu z wielokrotnymi zmiennymi opóźnieniami czasowymi. Rozwiązano problem optymalnego sterowania brzegowego dla systemów parabolicznych nieskończonego rzędu, w których wielokrotne zmienne opóźnienia czasowe występują zarówno w równaniach stanu oraz w warunkach brzegowych typu Neumanna. Tego rodzaju równania stanowią w liniowym przybliżeniu uniwersalny model matematyczny dla procesów dyfuzyjnych. Korzystając z metody Dubowickiego-Milutina wyprowadzono warunki konieczne i wystarczające optymalności dla problemu liniowo-kwadratowego.

Słowa kluczowe: sterowanie brzegowe, systemy paraboliczne nieskończonego rzędu, wielokrotne zmienne opóźnienia czasowe

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